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Maximal Paths on Rectangular Boards

Abstract: A combinatorial approach is made to the problem of obtaining a path on a rectangular board of m by n squares with both terminals at the edges of the board. A square is said to be covered **when the path enters one edge and leaves an adjacent edge. All other squares are said to be** missed. **Maximal paths are found, i.e., those which cover a maximum number of squares. For $m = n$, $m - 2$ squares are missed when m is even, and $m - 1$ squares are missed when m is odd. For $m < n$, $m - 2$ squares are missed if m is even, and $n - 2$ squares are missed if m is odd. The method of proof for m and n even is quite different from that for m or n odd. Certain properties of terminal positions, path length, types of missed squares, and unique paths are also investigated. The dependence of the results on the parity of m and n is again very striking.**

Introduction

In a systematic design procedure, the following problem arose: arrange a set of elements on a rectangular plate so that a maximum number of elements may be placed on the plate. All elements are assumed to be of equal size and to occupy a square area on the plate. If the input to the element is on one edge of the square, the output must be on an adjacent edge, perpendicular to the input. The set of elements is connected in series, i.e., the output of the first element is the input for the second, et cetera.

We formulate this problem as the determination of a path on a board of m by n squares with both terminals at the edges of the board. We call a square *covered* (i.e., an element can be placed in the square) whenever the path enters one edge of the square and leaves an adjacent edge. All other squares are said to be *missed.*

We find paths which cover the maximum number of squares (maximal paths), and investigate resulting properties of terminal positions, path length and types of missed squares. In certain cases maximal paths of minimum length having a maximum number of blank squares are unique.

The problem

Given a rectangular board composed of m rows and n columns of squares, we wish to describe a path through the squares of the board which satisfies the following properties.

(a) Both ends of the path must lie on the edges of the board, and no part of the path may be external to the board.

(b) If the path enters one edge of a square and leaves an adjacent edge, then this square cannot contain any other part of the path, and the square is said to be *covered* by the path.

(c) If the path enters one edge of a square and leaves the opposite edge, then the path may pass through this square perpendicular to the original direction. Such squares are said to be *missed* by the path, as well as all squares not entered by the path. Missed squares are classified into the following three types:

(0) Squares not entered by the path, called blanks: □,

(1) Squares having the path through 2 opposite sides; ⊟, and

(2) Squares having the path pass through in both directions; ⊞.

If a square has a Type 1 miss, then the path through the square is called a *wasted stroke*, either horizontal or vertical. A Type 2 miss has two wasted strokes.

(d) The desired path, called a *maximal path*, is to cover a maximum number of squares in the board.

Figure 1 shows a maximal path on a 3 × 5 board containing one missed square of each type. The missed squares are shaded.

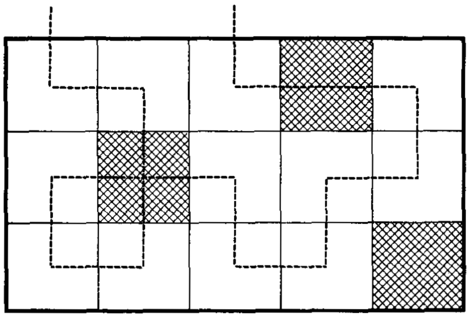


Figure 1
A maximal path on a 3 x 5 board.

Maximal path theorems and algorithms

The maximal paths must miss a minimum number of squares on the board. The following theorems determine the minimum number of misses on a board with m rows and n columns, for any m and n.

Each segment of the path, except the terminal segments, joins the centers of a pair of covered squares, horizontally or vertically. A terminal segment reaches the center of just one covered square. It follows from this pairing that if the row length is odd, each row has a missed square or contains a terminal segment. Now consider three cases, depending on the parity of m and n.

• *Theorem 1*

If m is even and n is odd, then any maximal path misses $m - 2$ squares and has horizontal terminal segments.

Proof. There are m rows of n squares each, and a square is missed in each row except those with horizontal terminal segments. Thus any path misses at least $m - 2$ squares and any path missing $m - 2$ has horizontal terminal segments. To construct a maximal path, proceed zigzag across the first pair of rows, turn and proceed back across the next pair of rows, et cetera. See Fig. 2.

• *Theorem 2*

If m and n are even, $m \leq n$, then any maximal path misses $m - 2$ squares and has parallel (horizontal if $m < n$) terminal segments.

Proof. There are less than two squares missed in a given row only if either (a) one square is missed and a terminal segment lies on the row, (b) both terminal segments lie on the row, or (c) the squares are paired 1-2, 3-4, 5-6, et cetera, by path segments. Now designate the top two rows as the first couple, the next two rows as the second couple, et cetera. Suppose some path misses less than $m - 2$ squares. Then condition (c) holds in both rows of some couple, since otherwise there are at least two squares missed in each couple except two, and at least two missed in these, for a total of at least $m - 2$ missed. By the same argument, condition (c) exists in both columns of some couple of columns. The four squares common to these two couples have the closed loop of Fig. 3, which is impossible.

If exactly $m - 2$ squares are missed, then condition (c) holds for one (exactly one if $m < n$)

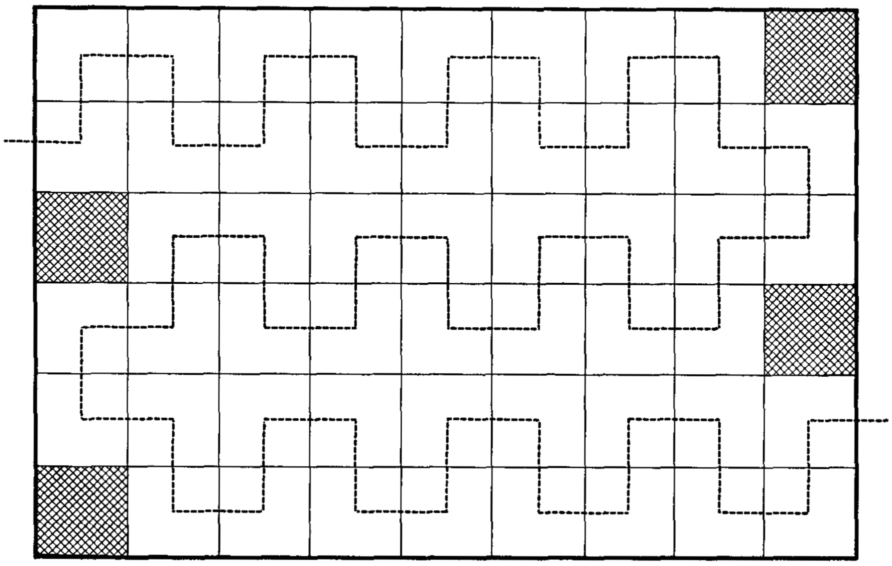


Figure 2
6 x 9 board maximal path with all missed squares blank.

row of each couple. Also, either condition (b) holds for some row (or column if $m = n$) or condition (a) holds for rows (or columns if $m = n$) in each of two couples. This proves the part about terminal segments. A maximal path can be constructed by zigzagging as in Fig. 2, but using Type 1 misses instead of blanks.

• *Theorem 3*

If m and n are odd, $m < n$, then any maximal path misses $n - 2$ squares and has vertical terminal segments. If $m = n$, a maximal path misses $n - 1$ squares and has perpendicular terminal segments.

Proof. Any path misses at least $n - 2$ squares, since a square is missed in each column except those with vertical terminal segments. If the path misses just $n - 2$, the missed squares then must be distributed with an odd number in each row. So when $m = n$, n squares will be missed if the terminal segments are parallel. If we have a horizontal and a vertical terminal segment, then we need miss only $n - 1$ squares. Maximal paths when $m = n$ are depicted in Fig. 4 with systematic extensions by adjoining L-shaped sections. When $m < n$ start with the array of Fig. 4 with $m + 2$ rows and $m + 2$ columns. Remove the two rows added last. Cut the figure vertically along AB. Insert $n - m - 2$ columns and connect the path cuts by horizontal zigzags.

The results of Theorems 1, 2, and 3 are summarized in Table 1.

Additional properties of maximal paths

The following theorems and remarks give additional properties of terminal locations for maximal paths, and also find maximal paths which have minimum length and which contain a maximum number of blanks. Some properties of uniqueness obtain under these conditions. We consider the following three cases; m and n even, m and n odd, and m even, n odd.

Table 1 **Maximal paths on a board with m rows and n columns.**

	$m = n$		$m < n$	
	m even	m odd	m even	m odd
No. of missed squares	$m - 2$	$m - 1$	$m - 2$	$n - 2$
Position of terminal segments	parallel	perpend.	horiz.	vertical

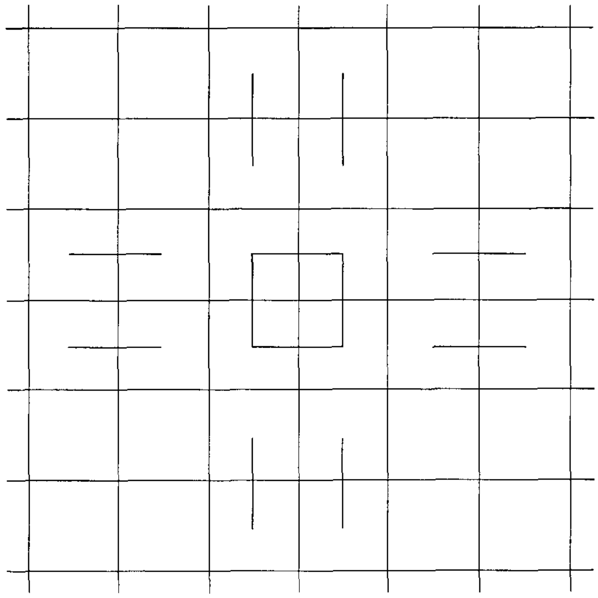


Figure 3 **A closed loop.**

• *Theorem 4*

Suppose m and n are even, $m \leq n$, and an m by n board contains a path missing $m - 2$ squares. Then the terminal segments are on the same side if and only if m is divisible by 4.

Proof. We know from Theorem 2 that the terminal segments are parallel and we may assume without loss that they are horizontal. Designate Columns 1 and 2 as the first couple, Columns 3 and 4 as the second couple, et cetera. Then in one of the $n/2$ couples of columns there are no missed squares, since all missed squares come in vertical pairs, and $2(n/2) > m - 2$. Think of the couple of columns in which there are no misses as a barrier. Now recall from the proof of Theorem 2 that condition (c) holds for one row of each couple of rows. In this row the path crosses the barrier. It cannot cross on the other row of the couple, since this would produce a closed loop. Thus, the barrier is crossed by the path exactly $m/2$ times, proving the theorem.

When m and n are even and $n \geq m > 4$, not all the missed squares can be blanks. The proof of this fact is straightforward, but tedious, and we omit it.

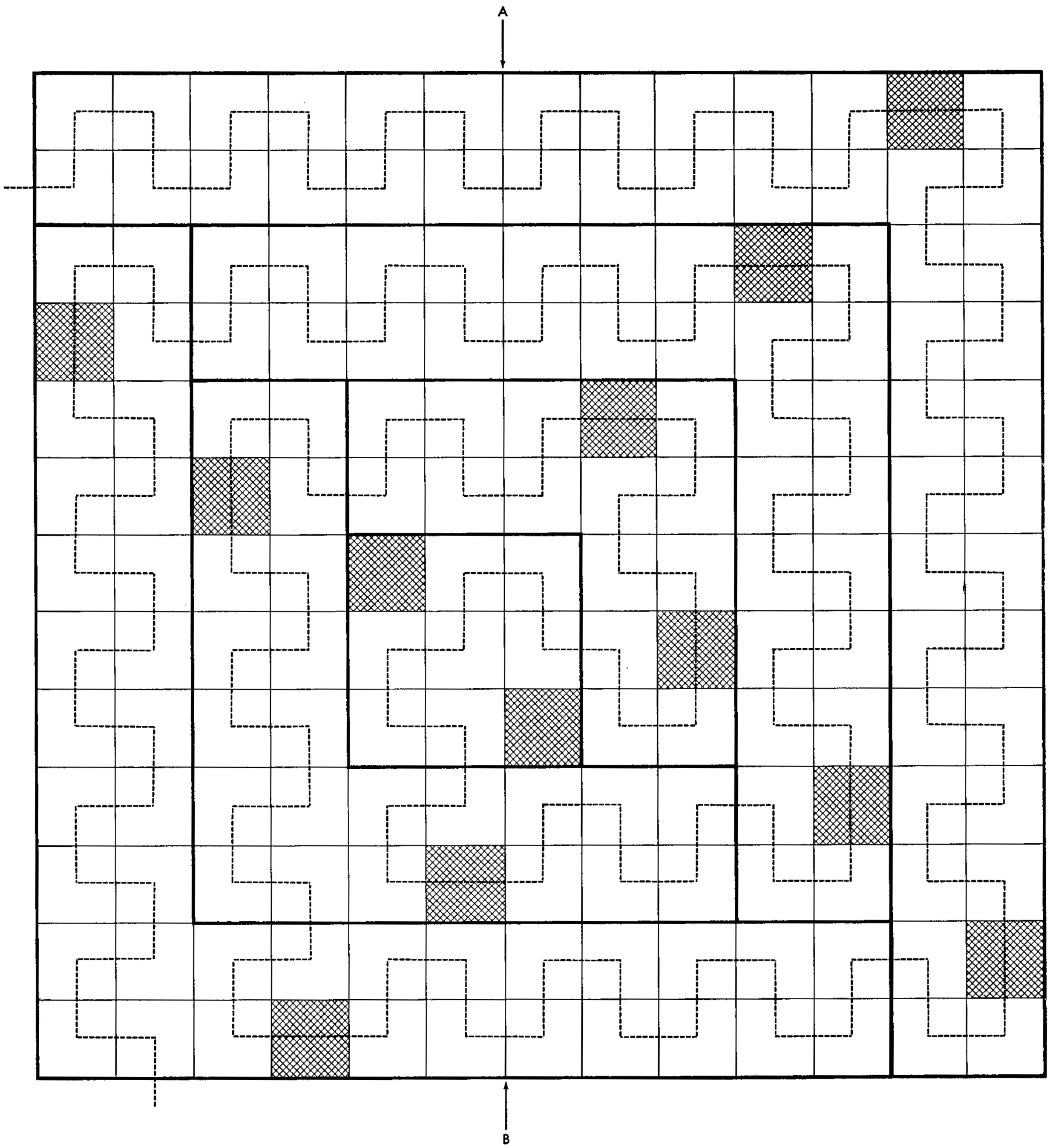


Figure 4 **Maximal paths on a square board where *m* is odd.**

Figure 5 gives maximal paths on square, even-sided boards having no wasted strokes when $m \leq 4$, and only one wasted stroke when $m > 4$. These maximal paths are thus minimal length and have a maximum number of blanks. To produce analogous paths on m by n rectangular boards, $m < n$, with m and n even, start with the m by m array of Fig. 5, cut at AB and add $n - m$ columns of m rows. Then connect the path with horizontal zigzags.

Figure 6 shows maximal paths on square boards where $m \equiv 2 \bmod 4$, in which both terminals are on the same row. Cutting along AB, adding $m - n$ columns, and connecting with horizontal zigzags produce maximal paths for any even $n \geq m$.

Note that in Fig. 5 when $m \equiv 0 \bmod 4$, $m > 4$, the terminals are separated by only one square, and for $m \equiv 2 \bmod 4$, $m > 2$, one terminal is located on the top row, and the other on the bottom row. For the zigzag path described in Theorems 1 and 2 the terminals may be located on the top and bottom rows for all m. These terminal locations give the maximum possible separation between terminals.

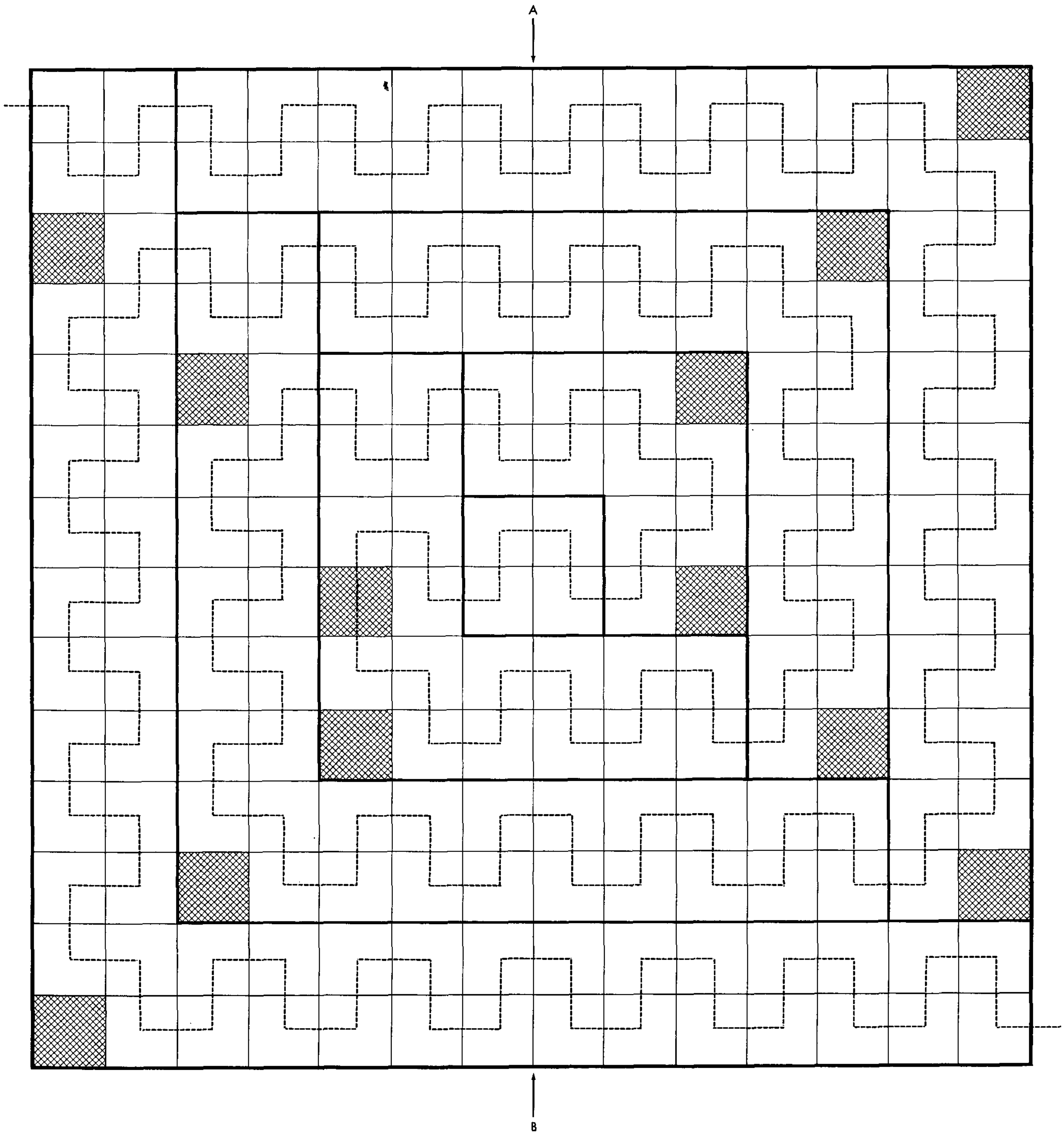
• *Theorem 5*

If m and n are odd, $m < n$, then any maximal path has at least $\frac{1}{2}(m - 3)$ wasted horizontal strokes and at least $\frac{1}{2}(m - 1)$ wasted vertical strokes.

Proof. Consider an even-numbered row. It contains a missed square. This is the only missed square in its column. The other squares in this column are paired by vertical path segments, and the square next above must be paired with the square next below by a wasted vertical stroke passing through this missed square. Hence there are $\frac{1}{2}(m - 1)$ wasted vertical strokes.

In any row the k blank squares occurring in even-numbered columns separate the row into $k + 1$ rowlets, each rowlet having an odd number of

Figure 5 **Maximal paths on square boards when m is even with $m - 3$ blank squares for $m > 4$.**

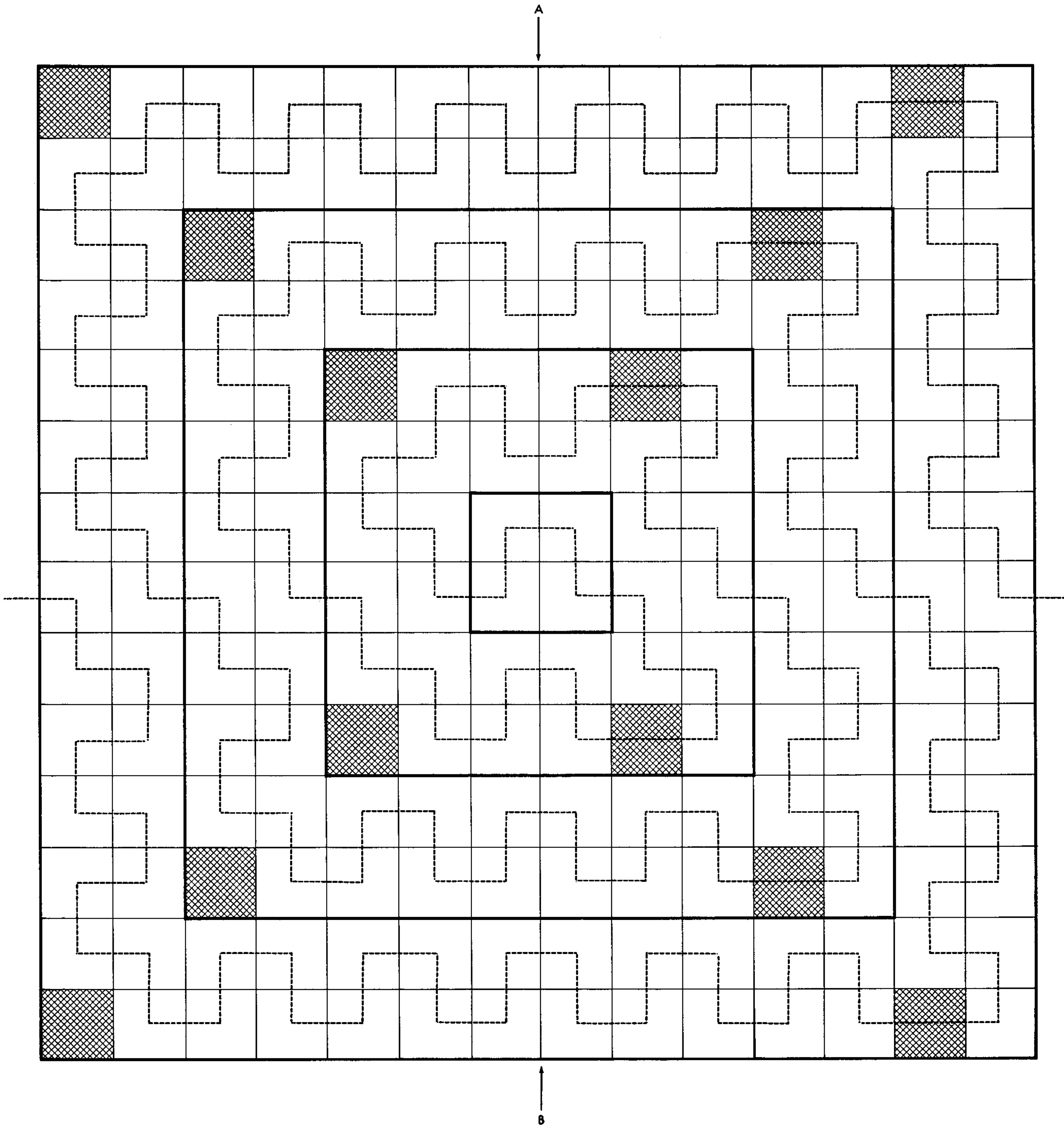


squares and hence a missed square. Only $\frac{1}{2}(n+1)$ of these missed squares can be in odd-numbered columns. Let x be the number of blank squares occurring in even-numbered columns. Then there are $m+x$ rowlets and at least $m+x-\frac{1}{2}(n+1)$ wasted horizontal strokes. There are at least $\frac{1}{2}(n-1)-2$ even-numbered columns which do not contain terminal segments. Hence there are at least $\frac{1}{2}(n-1)-2-x$ wasted horizontal strokes. Adding, we get $2h \geq m-3$, where h is the number of wasted horizontal strokes.

Figure 4, and the method of obtaining rectangular boards, gives maximal paths with minimum length for m and n odd, $m \leq n$. If $m = n$, then, by the first argument in the proof of Theorem 5, there are $\frac{1}{2}(m-3)$ wasted horizontal strokes and $\frac{1}{2}(m-3)$ wasted vertical strokes.

To obtain a maximal path of minimum length

Figure 6 **Maximal paths on square boards, $m \equiv 2 \bmod 4$, in which both terminals appear on the same row.**



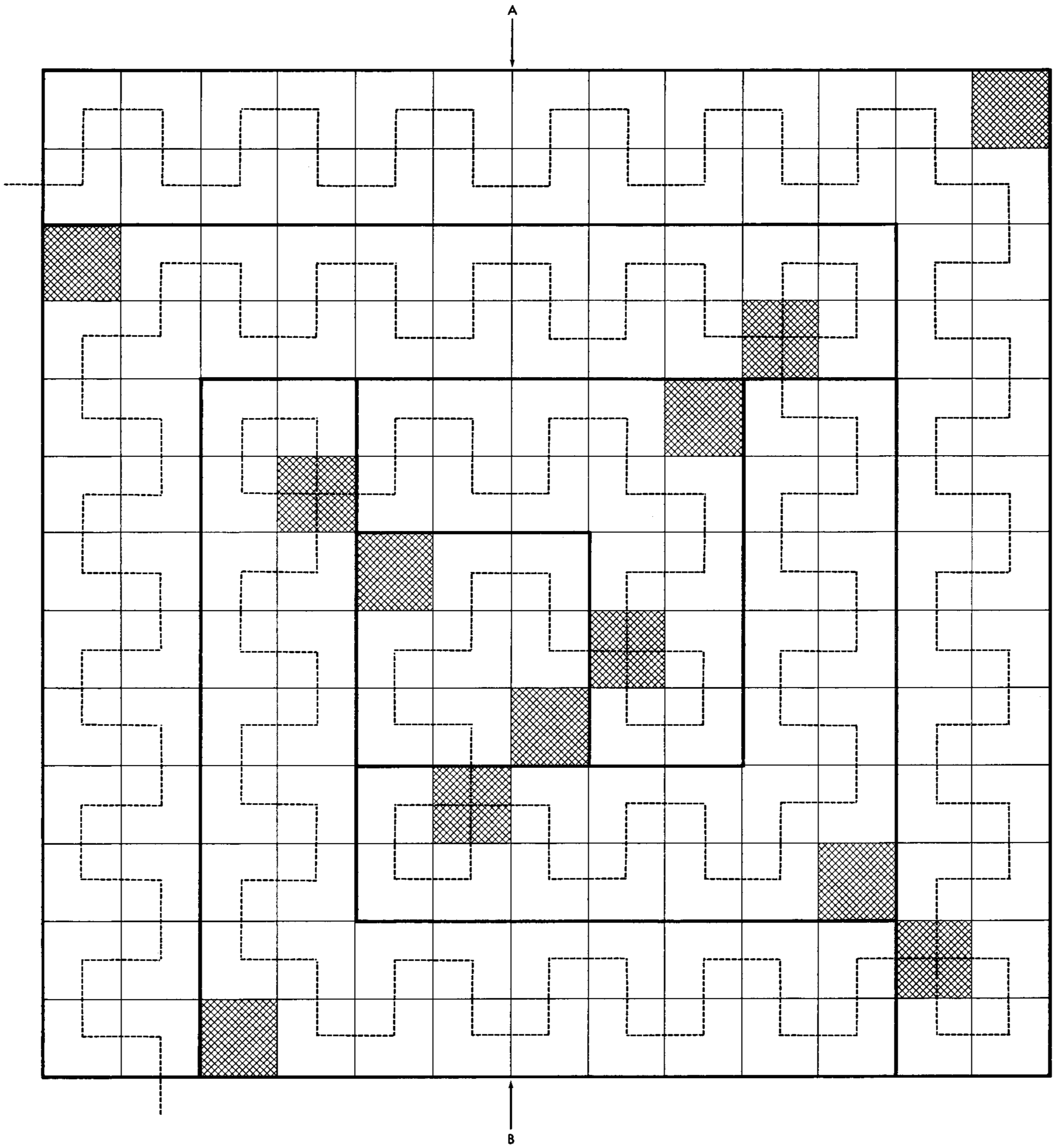


Figure 7 **Maximal paths of minimum length having a maximum number of blanks on square boards where m is odd.**

which has a maximum number of blanks when m and n are odd, misses of Type 2 must be used. Fig. 7 shows such paths. When $m = n$ there are $\frac{1}{2}(m - 3)$ Type 2 misses and $\frac{1}{2}(m + 1)$ blanks. To form a rectangular board, $m < n$, take the m by m board, add two columns with a vertical zigzag path having one wasted vertical stroke, and cut along AB adding $n - m - 2$ columns with horizontal zigzag paths.

If $m = n$ and m is odd, then the only maximal path having $\frac{1}{2}(m + 1)$ blanks is shown in Fig. 7, except for rotations and reflections. The idea of the proof of this fact is to detail the path structure in the outer pair of rows and columns. Show that a terminal lies on the second row from the edge, then that the path must zigzag across and down as in Fig. 7, and complete the argument by induction. Details are omitted.

For m even and n odd, the algorithm in the proof of Theorem 1 gives a maximal path with all $m - 2$

Table 2 **Additional properties of maximal paths.**

m and n even $m \leq n$	$m \equiv 0 \bmod 4$	(1) Terminals are on the same side. (2) Minimum length path has one wasted stroke, $m > 4$. See Fig. 5.
	$m \equiv 2 \bmod 4$	(1) Terminals are on opposite sides. (2) Terminals may appear on the same row. See Fig. 6. (3) Minimum length path has one wasted stroke, $m > 2$, and terminals can be at diagonal corners. See Fig. 5.
m and n odd	$m = n$	(1) Minimum length path has $\frac{1}{2}(m - 3)$ wasted strokes in each direction. (2) Minimum length path having a maximum number of blanks uses $\frac{1}{2}(m - 3)$ Type 2 misses and $\frac{1}{2}(m + 1)$ blanks. The path is unique. See Fig. 7. (3) Minimum length path having no Type 2 misses uses $m - 3$ Type 1 misses and 2 blanks. See Fig. 4.
	$m < n$	(1) Minimum length path has $\frac{1}{2}(m - 3)$ wasted horizontal strokes and $\frac{1}{2}(m - 1)$ wasted vertical strokes. (2) Minimum length path having a maximum number of blanks uses $\frac{1}{2}(m - 3)$ Type 2 misses, one Type 1 miss having a wasted vertical stroke, and $n - \frac{1}{2}(m + 3)$ blanks. Cut Fig. 7. (3) Minimum length path having no Type 2 misses uses $m - 2$ Type 1 misses and $n - m$ blanks. Cut Fig. 4.
m even n odd		(1) Maximum path of minimum length has blanks for all missed squares. (2) This path is unique when $n \geq 5$. See Fig. 2.

missed squares blank. Figure 2 shows such a path for a 6×9 board.

These maximal paths are obviously of minimum length and have a maximum number of blanks. It can be shown for $n \geq 5$ that this path is the unique maximal path of minimum length except for rotations and reflections.

A summary of the results in this section is shown in Table 2.

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