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Determining Component Variation for Gradual Transition between Dissimilar Impedances

Introduction

This note is an extension of previous work[1,2] and provides a derivation of impedance parameters involving application of continued fraction techniques. It also provides computed parameter values both in numeric and graphic form.

Applications of broad-band impedance-matching techniques presently enjoy an extended range of technical interest. In line terminations and couplings, components can be arrayed so as to provide gradual transitions between circuits and lines having dissimilar impedance characteristics. Optimized shock reflection conditions and energy flow result. Examples for application of gradual transition components are found in non-reflective terminations to electrical lines,[3] and in microwave phenomena,[4] optics,[5] and acoustics.[6]

Gradual transitions are characterized, in general, by non-linear differential equations of the Riccati type. Often, however, the numerous parameters are related to an insufficient number of boundary conditions to permit specific definition. Thereby, an intractable barrier is presented to determining optimum forms in which to realize the transitions.

The values presented in this paper are for the reactive lumped components of an analog gradual transition circuit. The circuit chosen specifically is a driving-point impedance with a dissipative, wide-frequency band.

• *Symbols*

f_a	fundamental frequency
f_b	frequency interval between successive zeros and poles
m, n	reactance ratios
R, L, C	lumped resistance, inductance and capacitance, respectively
x, y	resistance ratios
$Z=Z_r+Z_i$	complex impedance
Z_r, Z_i	real and reactive parts, respectively, of complex impedance
ξ	f_b/f_a
ζ	$2\pi f_a(LC)^{1/2}$

Features of termination

The driving-point impedance under consideration comprises, in general, the ladder network array of lumped components shown in Fig. 1. The input impedance Z has real part Z_r and reactive part Z_i. Condition requirements are that, over an appreciable and specified frequency range, Z_r should remain at a value closely approximating the characteristic line impedance, and Z_i should remain at or near zero in value. These requirements are met with suitable selection of circuit parameters.

Directives for selecting suitable circuit parameter values have been established and described elsewhere,[1,2] and will be outlined briefly here. If $R=0$, $Z_r=0$ and $1 \geqslant n_1 \geqslant n_2 \geqslant$----$n_n$, Z will vary with frequency as shown in Fig. 2. For optimization, the zero-to-pole frequency intervals, $f_{b1} \simeq f_{b2} \simeq f_{b3}$----$f_{b4}$, thus yielding the symmetrical characteristic shown in Fig. 3. When $R \neq 0$, it can be shown that, for variation in excitation frequency, the shunt resistances will be more effective in limiting variation in energy distribution betwen successive circuit loops than will the series resistances. Hence, $x=0$, $y=1$, and Z_r is determined by the magnitude of R, as shown in Fig. 4. Values of R are usually not critical but will lie in a range which can be determined conveniently in relation to the form of analogous circuit to be realized.

Of particular interest is derivation of the reactive component parameters, $n_1 n_2$----etc. and m_1, m_2----etc., to meet the equal zero-to-pole frequency interval requirement. Added versatility is afforded by allowing for variation in the frequency interval spacing ratio; $\xi = f_b/f_a$ extended to a large number of degrees of freedom.

Derivation of reactive component parameters

Using conventional methods[7] of circuit analysis, an LC network of the type under consideration may be represented in a continued fraction form. The fraction may be taken as a function of the circuit component values, or a function of the frequencies at which the impedance is zero or infinite. The problem to be solved is to define component values in terms of specified frequencies.

A moderately straightforward approach towards solution is the expansion of the impedance, as a function of frequency, to a continued fraction form by the process of synthetic division. However, the impracticality of this approach becomes evident when high degrees of freedom

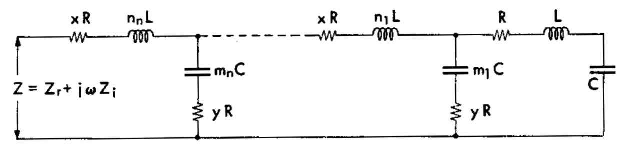

Figure 1 **Basic circuit.**

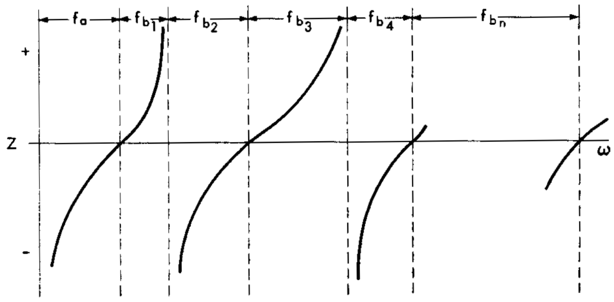

Figure 2 **Asymmetrical impedance characteristic.**

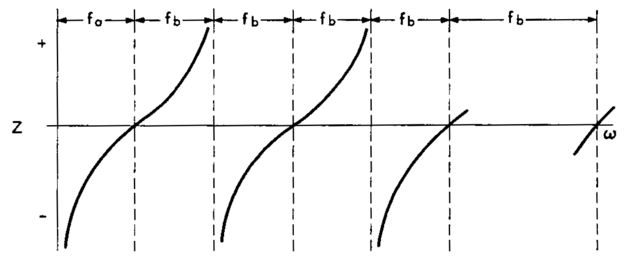

Figure 3 **Symmetrical impedance characteristic.**

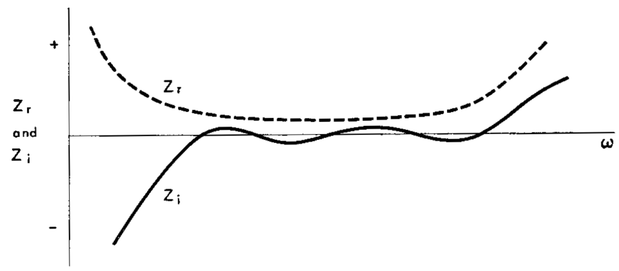

Figure 4 **Typical complex circuit characteristic (x=0), (y=1).**

are desired. The magnitude of the algebraic manipulation becomes formidable. Likewise, the numeric precision which must be maintained makes even the application of computing devices impractical. To alleviate these difficulties, a method of attack, although its basis lies in the division process, has been devised which is recursive in nature and demands a minimum of numeric precision.

The derivation proceeds as follows:
If we define; a_i as the ith pole of the network impedance
b_i as the residue of the ith pole,

we may then define the symbol, $A_k = \sum_{i=1}^{n} a_i^{2(k-1)} b_i$,

where n is equal to the degrees of freedom assigned to the network and $1<k<n$. Also,

$a_i^{2(k-1)} = 1$, if $k=1$ and $a_1=0$

$a_i^{2(k-1)} = 0$, if $k=1$ and $a_2=0$.

Representing the impedance Z as a function of $\omega = 2\pi f$, we have the rational form;

$$Z = \frac{K(\omega^2+1)[\omega^2+(1+2\xi)^2 \text{ --- } \{\omega^2+[1+2(n-1)\xi]^2\}}{\omega[\omega^2+(1+\xi)^2 \text{ ---- } \{\omega^2+[1+(2n-3)\xi]^2\}}$$

Normalizing Z, so that $K=1$, and using the previous definition of, $\xi = f_b/f_a$, we may then proceed to expand Z as the ratio of two polynomials, defining X_i as the i th coefficient in the denominator of Z.

Thus:

$$Z = \frac{\omega^{2n}+(A_1+X_1)\omega^{2n-2}+(A_1X_1-A_2+X_2)\omega^{2n-4}+(A_1X_2-A_2X_1+A_3+X_3)\omega^{2n-6}+--}{\omega^{2n-1}+X_1\omega^{2n-3}+X_2\omega^{2n-5}+X_3\omega^{2n-7}+------}$$

Performing synthetic division on the above representation of Z, we obtain the variables, here showing the n th terms:

$$M_n = \frac{A_{n+1}}{A_n}$$

$$N_{n-1} = A_1\frac{M_n}{M_1} - A_n = A_1\left(\frac{M_n}{M_1} - \frac{A_n}{A_1}\right)$$

$$O_{n-2} = M_1\frac{N_{n-1}}{N_1} - M_{n-1} = \left(\frac{N_{n-1}}{N_1} - \frac{M_{n-1}}{M_1}\right)$$

etc.

From this, the results of the synthetic division may be expressed as follows;

$$L_n = 1\,, \qquad C_n = \frac{1}{A_1}\,,$$

$$L_{n-1} = \frac{A_1}{M_1}, \qquad C_{n-1} = \frac{N_1}{O_1}\,,$$

$$L_{n-2} = \frac{O_1}{P_1}, \qquad C_{n-2} = \frac{P_1}{Q_1}\,,$$

continued 2_n times.

The parameters derived are then represented in normalized form as;

$$n_n = \frac{1}{L_1}, \qquad m_n = \frac{C_n}{C_1},$$

$$n_{n-1} = \frac{L_{n-1}}{L_1}, \qquad m_{n-1} = \frac{C_{n-1}}{C_n},$$

$$n_{n-2} = \frac{L_{n-2}}{L_1}, \qquad m_{n-2} = \frac{C_{n-2}}{C_n},$$

$$\vdots \qquad\qquad \vdots$$

$$n_1 = 1, \qquad\qquad m_1 = 1,$$

with the $2n+1$ parameter defined as $\zeta = \sqrt{L_1 C_1}$.

Tabulated and representative graphical data,[8] evaluated on an IBM 704, follow in Tables 1 through 8 and Figs. 8 and 9; these give values of circuit parameters for representative values of ξ.

Illustration of transition circuit realization

Application of the circuit parameter relationships derived above is illustrated in the realization of an acoustical circuit.[1] This circuit, of limited thickness, forms a transition between a ρc medium and a plane reflecting surface. Acoustical absorption is required over a wide frequency band, and reactive circuit components are introduced to extend the range of effective absorption into the low frequencies.

With six degrees of freedom required to cover the frequency band of high absorption, the basic circuit is derived from Fig. 1, setting $x=0$, suffix $n=6$, and obtaining respective n and m values from Table 4 or Fig. 8. An analogous form of acoustical circuit, having series inertances and shunt tanks and resistances is shown in Fig. 5. Limiting the thickness to three inches, internal dimensions are obtained from:

$$\text{Hole inertance, } L \approx \frac{\rho \text{ (hole length)}}{\pi \text{ (hole radius)}^2}$$

and

$$\text{Tank capacitance, } C \approx \frac{\text{(tank volume)}}{\rho c^2}.$$

Values for the shunt resistances are not critical, and may be selected most conveniently in relation to measured characteristics. In this example, the shunt resistances are provided by porous material having a specific acoustical resistance of approximately 300 rayls/cm., placed to fill the tanks. At the primary surface, the transition presents an array of 29/64 inch holes at one inch centers in a square pattern.

The analogous acoustical circuit approaches most closely the pure reactance case with all porous material removed. The respective normal specific acoustical impedance characteristic is shown in Fig. 6. Suitable dimen-

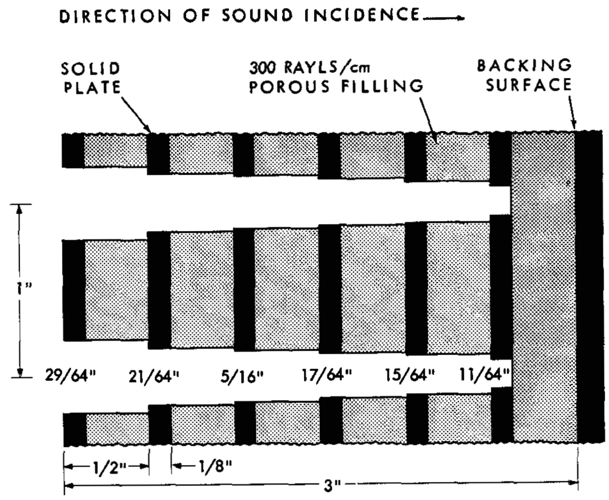

Figure 5 **Form of acoustical circuit.**

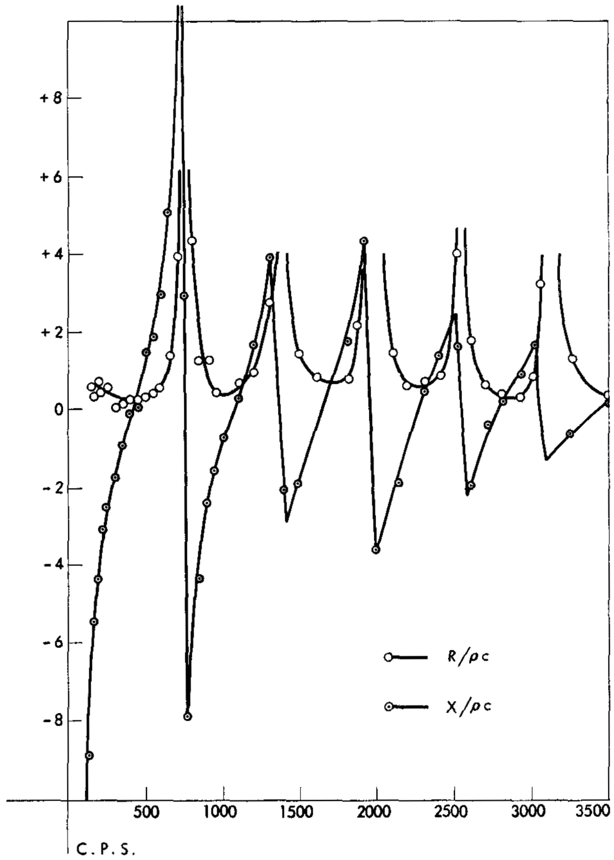

Figure 6 **Measured normal specific acoustical impedance characteristic of circuit with porous material removed.**

sioning of the reactive components has resulted in uniform frequency intervals between the successive zeros.

The reactive circuit framework is now in a form suitable for directing energy for dissipation in shunt resistances uniformly over a wide frequency band. Addition of shunt resistances to the acoustical circuit results in the impedance characteristics shown in Fig. 7. These impedance values correspond to uniform broad-band sound absorption.

References

1. C. Becker, *Ac. Soc. Am. J.*, **26,** 798 (1954).
2. C. Becker, *Ac. Soc. Am. J.*, **28,** 1068 (1956).
3. L. R. Walker and N. Wax, *J. Appl. Phys.* **17,** 1043 (1946).
4. W. Kofink, *Ann. Phys.* **1,** 119 (1947).
5. W. Geffcken, *Ann. Phys.* **40,** 385 (1941).
6. A. Schoch, *Noise and Sound Transmission,* Report of the Physical Society, 167 (1949).
7. W. Cauer, *Synthesis of Linear Communication Networks,* McGraw-Hill 1958, Chapter 5.
8. Complete graphical data, corresponding to all tabulated data, are available from the authors.

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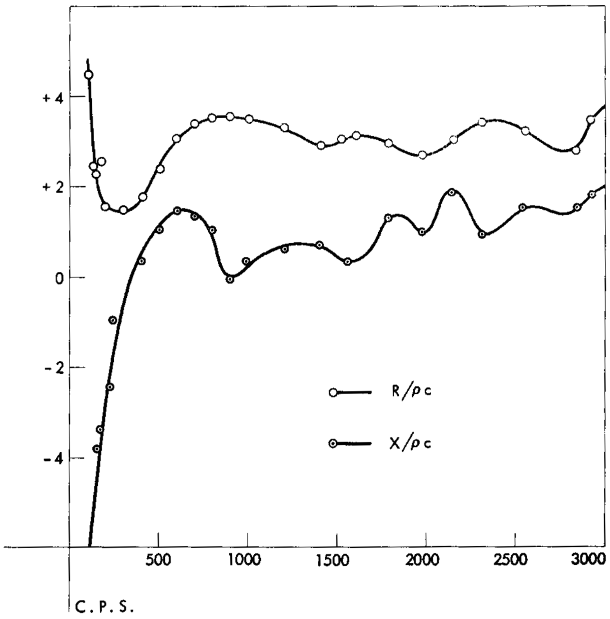

Figure 7 **Measured normal specific acoustical impedance characteristic of circuit with porous material in place.**

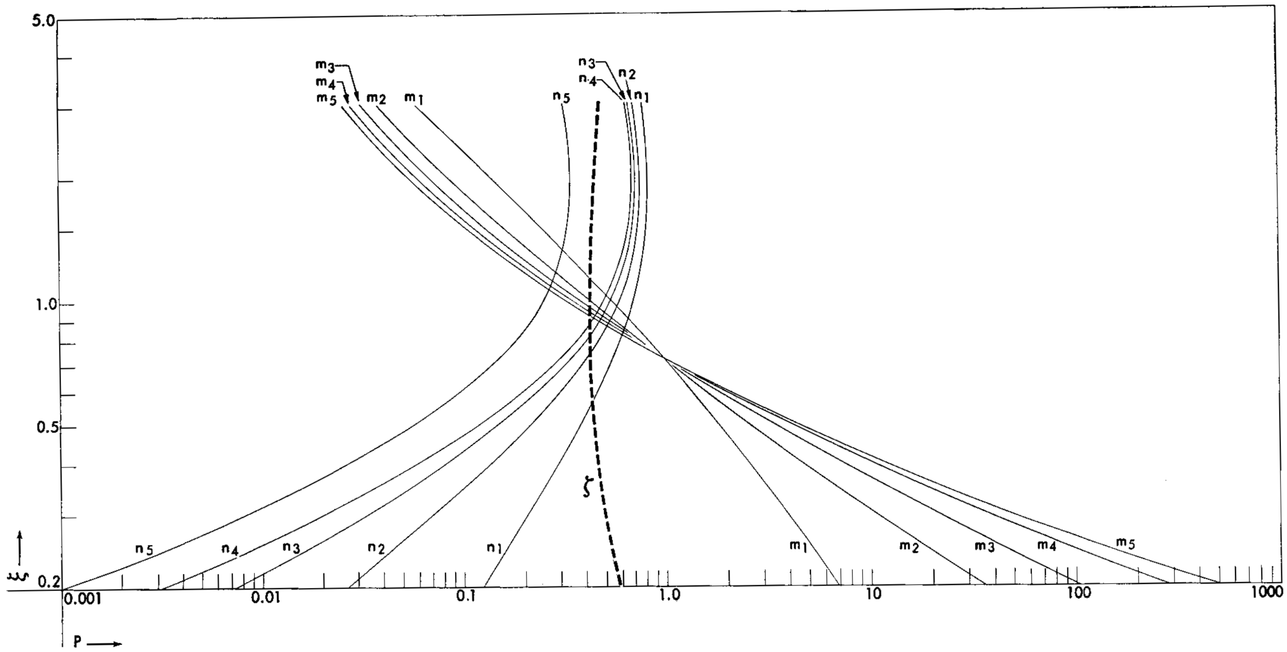

Figure 8 **Circuit parameters (P) for 6 degrees of freedom.**

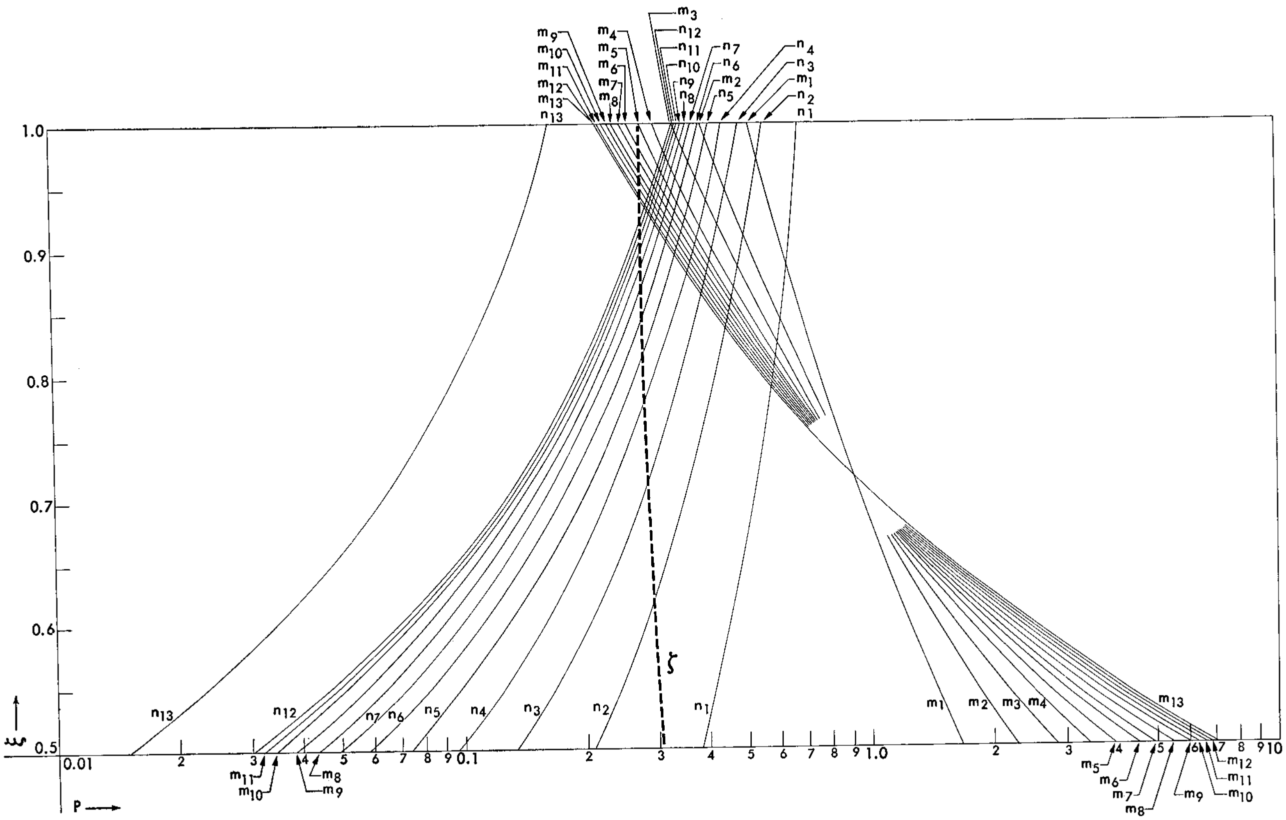

Figure 9 **Circuit parameters (*P*) for 14 degrees of freedom.**

Table 1 **Values for 3 degrees of freedom.**

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
n_1	0.104	0.205	0.315	0.422	0.517	0.597	0.661	0.712	0.750	0.833	0.837	0.799
n_2	0.0201	0.0584	0.110	0.165	0.217	0.262	0.298	0.327	0.350	0.399	0.402	0.380
m_1	9.64	4.71	2.83	1.90	1.38	1.05	0.826	0.670	0.556	0.270	0.161	0.0777
m_2	38.4	10.8	4.67	2.53	1.57	1.07	0.784	0.600	0.476	0.208	0.120	0.0570
ζ	0.765	0.712	0.681	0.660	0.647	0.637	0.634	0.632	0.633	0.639	0.650	0.667

Table 2 **Values for 4 degrees of freedom.**

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
n_1	0.108	0.196	0.292	0.390	0.480	0.560	0.626	0.680	0.722	0.817	0.824	0.784
n_2	0.0230	0.0736	0.152	0.247	0.345	0.437	0.517	0.583	0.636	0.756	0.766	0.719
n_3	0.00611	0.0259	0.0613	0.107	0.156	0.203	0.244	0.279	0.306	0.370	0.375	0.381
m_1	8.46	4.40	2.74	1.87	1.36	1.03	0.811	0.654	0.538	0.253	0.148	0.0688
m_2	41.7	11.9	5.03	2.64	1.60	1.06	0.755	0.566	0.440	0.181	0.101	0.0465
m_3	112.	20.6	6.85	3.12	1.72	1.08	0.734	0.535	0.408	0.160	0.0884	0.0405
ζ	0.699	0.640	0.594	0.568	0.552	0.543	0.537	0.535	0.535	0.542	0.557	0.580

Table 3 **Values for 5 degrees of freedom.**

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
n_1	0.117	0.201	0.290	0.379	0.465	0.543	0.609	0.664	0.708	0.811	0.819	0.778
n_2	0.0238	0.0687	0.137	0.222	0.314	0.403	0.483	0.552	0.607	0.739	0.751	0.702
n_3	0.00751	0.0348	0.0893	0.167	0.256	0.347	0.431	0.504	0.564	0.708	0.721	0.669
n_4	0.00238	0.0137	0.0387	0.0760	0.120	0.166	0.208	0.245	0.276	0.349	0.356	0.330
m_1	7.58	4.11	2.63	1.83	1.34	1.02	0.801	0.645	0.529	0.244	0.140	0.0634
m_2	35.6	11.1	4.90	2.62	1.58	1.05	0.738	0.548	0.424	0.168	0.0920	0.0413
m_3	121.	22.5	7.38	3.27	1.75	1.06	0.710	0.507	0.380	0.141	0.0763	0.0342
m_4	254.	33.2	9.09	3.64	1.84	1.07	0.699	0.490	0.363	0.131	0.0705	0.0316
ζ	0.641	0.575	0.533	0.507	0.491	0.482	0.476	0.473	0.471	0.481	0.499	0.523

Table 4 **Values for 6 degrees of freedom.**

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
n_1	0.125	0.206	0.291	0.376	0.458	0.534	0.600	0.655	0.700	0.807	0.817	0.775
n_2	0.0263	0.0700	0.133	0.213	0.300	0.386	0.465	0.534	0.591	0.731	0.744	0.694
n_3	0.00756	0.0319	0.0797	0.149	0.233	0.320	0.403	0.476	0.538	0.691	0.707	0.654
n_4	0.00308	0.0193	0.0587	0.122	0.203	0.290	0.374	0.449	0.512	0.672	0.689	0.634
n_5	0.00109	0.00813	0.0265	0.0573	0.0970	0.140	0.182	0.220	0.252	0.333	0.341	0.314
m_1	7.00	3.90	2.54	1.79	1.32	1.01	0.795	0.639	0.524	0.238	0.134	0.0598
m_2	30.6	10.2	4.68	2.56	1.56	1.03	0.727	0.538	0.414	0.160	0.0864	0.0381
m_3	104.	21.4	7.26	3.26	1.74	1.05	0.695	0.491	0.365	0.131	0.0696	0.0306
m_4	277.	36.5	9.77	3.80	1.86	1.06	0.677	0.466	0.340	0.117	0.0620	0.0273
m_5	493.	48.7	11.40	4.10	1.93	1.07	0.670	0.456	0.330	0.112	0.0589	0.0259
ζ	0.598	0.530	0.489	0.465	0.456	0.436	0.430	0.427	0.426	0.437	0.454	0.483

Table 5 **Values for 8 degrees of freedom.**

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
n_1	0.137	0.217	0.295	0.376	0.453	0.525	0.589	0.645	0.690	0.804	0.816	0.772
n_2	0.0314	0.0753	0.135	0.207	0.288	0.369	0.446	0.515	0.574	0.723	0.739	0.687
n_3	0.00929	0.0332	0.0762	0.138	0.213	0.295	0.375	0.449	0.512	0.677	0.696	0.640
n_4	0.00330	0.0174	0.0499	0.103	0.174	0.254	0.335	0.410	0.476	0.649	0.669	0.612
n_5	0.00141	0.0107	0.0371	0.0851	0.152	0.231	0.311	0.387	0.454	0.632	0.653	0.595

Table 5 (continued)

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
n_6	0.000784	0.00791	0.0311	0.0761	0.141	0.218	0.299	0.375	0.443	0.622	0.644	0.586
n_7	0.000315	0.00356	0.0146	0.0367	0.0688	0.107	0.147	0.186	0.219	0.310	0.321	0.291
m_1	6.30	3.63	2.42	1.73	1.295	0.997	0.786	0.632	0.517	0.230	0.127	0.0550
m_2	24.7	8.90	4.32	2.44	1.516	1.012	0.714	0.526	0.402	0.151	0.0796	0.0341
m_3	77.3	18.2	6.71	3.13	1.69	1.028	0.676	0.473	0.348	0.120	0.0622	0.0266
m_4	208.7	32.8	9.45	3.77	1.84	1.042	0.653	0.443	0.318	0.105	0.0535	0.0229
m_5	490.9	52.4	12.24	4.31	1.96	1.053	0.639	0.424	0.300	0.0956	0.0486	0.0208
m_6	955.9	73.3	14.6	4.71	2.03	1.060	0.631	0.413	0.290	0.0906	0.0459	0.0197
m_7	1395.	87.5	15.9	4.92	2.07	1.064	0.627	0.407	0.285	0.0883	0.0446	0.0192
ζ	0.532	0.467	0.428	0.403	0.385	0.375	0.368	0.365	0.365	0.366	0.395	0.425

Table 6 **Values for 10 degrees of freedom.**

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
n_1	0.145	0.225	0.302	0.377	0.451	0.520	0.584	0.639	0.685	0.803	0.816	0.772
n_2	0.0351	0.0801	0.138	0.207	0.283	0.361	0.437	0.505	0.564	0.719	0.738	0.683
n_3	0.0110	0.0357	0.0772	0.135	0.206	0.284	0.362	0.435	0.499	0.670	0.692	0.635
n_4	0.00404	0.0184	0.0491	0.0982	0.164	0.239	0.317	0.392	0.458	0.639	0.662	0.603
n_5	0.00167	0.0107	0.0344	0.0771	0.138	0.211	0.289	0.364	0.431	0.617	0.641	0.582
n_6	0.000780	0.00687	0.0262	0.0646	0.122	0.193	0.270	0.345	0.413	0.602	0.627	0.567
n_7	0.000421	0.00495	0.0216	0.0571	0.112	0.182	0.259	0.334	0.402	0.593	0.618	0.558
n_8	0.000277	0.00403	0.0192	0.0531	0.107	0.176	0.252	0.327	0.396	0.587	0.613	0.552
n_9	0.000119	0.00188	0.00924	0.0259	0.0527	0.0870	0.125	0.162	0.197	0.293	0.306	0.275
m_1	5.91	3.46	2.34	1.69	1.28	0.988	0.781	0.628	0.514	0.225	0.122	0.0517
m_2	21.6	8.12	4.07	2.35	1.48	0.997	0.705	0.519	0.396	0.146	0.0754	0.0316
m_3	62.8	16.0	6.22	2.99	1.65	1.009	0.664	0.464	0.340	0.114	0.0580	0.0243
m_4	160.	28.3	8.76	3.62	1.80	1.022	0.638	0.430	0.307	0.0978	0.0490	0.0206
m_5	368.	46.1	11.6	4.22	1.93	1.033	0.621	0.408	0.286	0.0877	0.0436	0.0183
m_6	775.	69.5	14.5	4.74	2.03	1.042	0.610	0.393	0.272	0.0812	0.0402	0.0169
m_7	1466.	96.3	17.3	5.17	2.10	1.049	0.602	0.383	0.263	0.0770	0.0380	0.0160
m_8	2378.	121.	19.4	5.47	2.16	1.054	0.597	0.376	0.257	0.0745	0.0367	0.0155
m_9	3104.	137.	20.5	5.63	2.18	1.056	0.595	0.374	0.254	0.0734	0.0361	0.0152
ζ	0.535	0.423	0.385	0.361	0.344	0.334	0.328	0.326	0.325	0.336	0.354	0.387

Table 7 **Values for 12 degrees of freedom.**

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
n_1	0.150	0.230	0.306	0.379	0.450	0.518	0.580	0.635	0.682	0.803	0.817	0.772
n_2	0.0400	0.0838	0.141	0.208	0.281	0.357	0.431	0.499	0.558	0.718	0.738	0.683
n_3	0.0123	0.0381	0.0790	0.134	0.203	0.278	0.354	0.426	0.491	0.667	0.690	0.632
n_4	0.00473	0.0198	0.0499	0.0967	0.159	0.231	0.307	0.381	0.448	0.634	0.659	0.599
n_5	0.00201	0.0114	0.0342	0.0744	0.132	0.201	0.277	0.351	0.418	0.610	0.636	0.575
n_6	0.000933	0.00708	0.0252	0.0605	0.114	0.181	0.255	0.329	0.397	0.592	0.619	0.558
n_7	0.000472	0.00475	0.0196	0.0514	0.102	0.167	0.240	0.314	0.382	0.580	0.607	0.546
n_8	0.000261	0.00345	0.0162	0.0455	0.0937	0.158	0.230	0.303	0.372	0.571	0.588	0.537
n_9	0.000163	0.00272	0.0142	0.0418	0.0885	0.151	0.223	0.296	0.365	0.564	0.593	0.531
n_{10}	0.000120	0.00235	0.0131	0.0397	0.0856	0.148	0.219	0.292	0.361	0.561	0.589	0.527
n_{11}	0.0000538	0.00112	0.00635	0.0195	0.0423	0.0733	0.109	0.146	0.180	0.280	0.294	0.263
m_1	5.65	3.35	2.28	1.66	1.262	0.981	0.777	0.626	0.511	0.222	0.119	0.0493
m_2	19.68	7.61	3.89	2.28	1.456	0.986	0.699	0.515	0.392	0.142	0.073	0.0298
m_3	54.62	14.6	5.86	2.89	1.621	0.995	0.656	0.458	0.335	0.111	0.055	0.0227
m_4	131.3	25.2	8.17	3.49	1.761	1.005	0.629	0.422	0.301	0.0940	0.046	0.0190
m_5	291.	40.4	10.84	4.07	1.890	1.015	0.610	0.398	0.278	0.0830	0.041	0.0168
m_6	595.	61.4	13.75	4.62	1.99	1.025	0.596	0.381	0.262	0.0759	0.037	0.0153
m_7	1150.	88.0	16.78	5.12	2.09	1.032	0.586	0.369	0.251	0.0710	0.034	0.0143
m_8	2065.	119.	19.7	5.56	2.16	1.039	0.579	0.360	0.242	0.0676	0.033	0.0136
m_9	3384.	152.	22.3	5.90	2.22	1.044	0.574	0.354	0.237	0.0653	0.032	0.0131
m_{10}	4879.	179.	24.2	6.14	2.25	1.047	0.571	0.350	0.233	0.0638	0.031	0.0128
m_{11}	5944.	195.	25.2	6.27	2.27	1.049	0.570	0.350	0.232	0.0631	0.030	0.0127
ζ	0.449	0.389	0.353	0.330	0.314	0.304	0.298	0.296	0.295	0.306	0.325	0.358

Table 8 **Values for 14 degrees of freedom.**

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0
n_1	0.154	0.234	0.309	0.381	0.450	0.517	0.578	0.633	0.679	0.803	0.818
n_2	0.0400	0.0866	0.144	0.209	0.281	0.355	0.427	0.495	0.554	0.717	0.738
n_3	0.0135	0.0403	0.0804	0.135	0.201	0.274	0.350	0.420	0.485	0.666	0.690
n_4	0.00532	0.0207	0.0518	0.0964	0.157	0.227	0.300	0.374	0.441	0.630	0.658
n_5	0.00228	0.0124	0.0342	0.0735	0.128	0.195	0.270	0.342	0.410	0.608	0.633
n_6	0.00114	0.00750	0.0254	0.0592	0.109	0.174	0.245	0.319	0.387	0.584	0.615
n_7	0.000552	0.00487	0.0195	0.0485	0.0974	0.159	0.229	0.302	0.370	0.573	0.604

Table 8 (continued)

parameter	$\xi \longrightarrow$ 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0
n_8	0.000302	0.00352	0.0152	0.0426	0.0865	0.147	0.218	0.289	0.358	0.562	0.590
n_9	0.000175	0.00254	0.0128	0.0377	0.0804	0.139	0.209	0.280	0.348	0.552	0.583
n_{10}	0.000109	0.00199	0.0111	0.0344	0.0759	0.134	0.202	0.273	0.342	0.546	0.577
n_{11}	0.0000753	0.00166	0.0100	0.0324	0.0727	0.130	0.198	0.269	0.337	0.542	0.573
n_{12}	0.0000594	0.00149	0.00945	0.0312	0.0710	0.128	0.195	0.266	0.334	0.540	0.571
n_{13}	0.0000273	0.000718	0.0046	0.0154	0.0352	0.0634	0.0971	0.133	0.167	0.269	0.284
m_1	5.47	3.27	2.24	1.64	1.25	0.976	0.774	0.624	0.509	0.219	0.116
m_2	18.4	7.25	3.76	2.23	1.44	0.977	0.694	0.512	0.389	0.140	0.0704
m_3	49.1	13.5	5.57	2.80	1.59	0.983	0.651	0.454	0.331	0.108	0.0532
m_4	115.	23.0	7.72	3.38	1.73	0.992	0.620	0.417	0.296	0.0905	0.0441
m_5	241.	35.5	10.30	3.91	1.86	1.004	0.602	0.393	0.272	0.0798	0.0386
m_6	480.	56.4	12.6	4.51	1.94	1.004	0.588	0.372	0.255	0.0725	0.0346
m_7	929.	76.1	16.3	4.96	2.07	1.022	0.574	0.361	0.243	0.0668	0.0322
m_8	1608.	108.	18.9	5.46	2.14	1.022	0.567	0.349	0.233	0.0636	0.0301
m_9	2798.	144.	21.9	5.93	2.20	1.030	0.562	0.342	0.227	0.0604	0.0287
m_{10}	4480.	180.	24.9	6.25	2.27	1.034	0.556	0.336	0.221	0.0584	0.0278
m_{11}	6630.	217.	27.2	6.55	2.31	1.038	0.552	0.332	0.218	0.0570	0.0270
m_{12}	8828.	246.	28.9	6.75	2.34	1.040	0.550	0.329	0.215	0.0561	0.0266
m_{13}	10276.	263.	29.8	6.85	2.35	1.042	0.549	0.327	0.214	0.0556	0.0264
ζ	0.420	0.362	0.326	0.305	0.291	0.281	0.276	0.273	0.272	0.273	0.302