

A Duality Theorem for Convex Programs*

Abstract: A proof is given for a duality theorem for a class of convex programs, i.e., constrained minimization of convex functions. A simple example is included.

1. Introduction

Linear programming is concerned with minimizing (or maximizing) a linear function of n variables subject to linear constraints on the variables. The constraints may be in the form of either equations or inequalities or both. Such mathematical problems arise in a variety of contexts both in the engineering sciences and in the business sciences.

For any linear program which has a finite solution there exists a companion problem called its *dual*. The dual problem also has a finite solution, and moreover, certain properties of the solution of the dual problem may be deduced immediately from the solution of the original program. This duality relationship has both theoretical and computational significance and has provided a valuable tool for investigators in this field.

In a recent paper¹ a duality relationship was established for a class of quadratic programs in which one wishes to minimize some quadratic function of n variables subject to linear constraints.

In what follows, this concept will be extended further to include minimization of a general convex function subject to linear constraints. The development, to some extent, parallels that given in the quadratic case. The notion of duality for this class of programs has also been discussed previously by Dennis.² The relationship of the present development to that of Dennis will be considered in Section 7.

2. Notation

Matrix notation will be used in this paper. Lower case letters $x, y, \dots$ will denote column vectors and capital letters $A, C, \dots$ will be matrices. The operator ∇ is the gradient operator and $\nabla f(x)$ is a vector composed of the first partial derivatives of the scalar function $f(x)$.

* This research was supported by the AEC Computing and Applied Mathematics Center, Institute of Mathematical Sciences, New York University, under Contract No. AT(30-1)-1480 with the U. S. Atomic Energy Commission. The work was done at NYU.

Prime denotes *transpose*, so that $x', y', \dots$ are row vectors. The product $x'y$ is the inner product of the vectors x and y . A vector inequality will apply to each component of the vector, i.e., $x \geq 0$ implies that each component of x is non-negative.

3. Convex functions

A scalar function, f , of an n -dimensional vector, x , is *convex* if linear interpolation does not underestimate the value of the function, i.e.,

$$f[x_1 + k(x_2 - x_1)] \leq f(x_1) + k[f(x_2) - f(x_1)] \quad (3.1)$$

for $0 \leq k \leq 1$. For convex functions a local minimum is also a global minimum.

For the proofs in the succeeding sections the following lemma regarding convex functions is required.

• Lemma

If $f(x)$ is convex and continuously differentiable, then for any vectors y and x

$$f(y) - f(x) \geq (y - x)' \nabla f(x). \quad (3.2)$$

• Proof

From the assumption of convexity

$$f[x + k(y - x)] \leq f(x) + k[f(y) - f(x)]$$

for all k such that $0 \leq k \leq 1$. Rearranging terms

$$k[f(y) - f(x)] \geq f[x + k(y - x)] - f(x). \quad (3.3)$$

From the differential mean value theorem,

$$f[x + k(y - x)] - f(x) = k(y - x)' \nabla f(x + x_0),$$

where $x_0 = \theta k(y - x)$ for some θ such that $0 \leq \theta \leq 1$.

Using this in the right-hand member of (3.3),

$$f(y) - f(x) \geq (y-x)' \nabla f(x+x_0).$$

Letting k approach zero, the inequality (3.2) results.

4. Duality in linear programming

The linear programming problem may be posed as follows: To minimize the linear form $p'x$ over all n -dimensional vectors x satisfying the constraints

$$Ax \geq b \quad (4.1)$$

$$x \geq 0, \quad (4.2)$$

where p is a $n \times 1$ vector, b is a $m \times 1$ vector and A is an $m \times n$ matrix.

The dual problem to the above is to maximize $b'v$ over all m -dimensional vectors v satisfying

$$A'v \leq p \quad (4.3)$$

$$v \geq 0. \quad (4.4)$$

The duality theorem³ states that if a solution to either problem exists and is finite, then a solution to the other problem also exists and indeed

$$\text{Minimum } p'x = \text{Maximum } b'v. \quad (4.5)$$

The constraints (4.1), (4.2) in the primal problem may take on other forms with corresponding changes in the dual constraints (4.3), (4.4). One form which will be of particular interest is given here. Let the vector x be partitioned as (x_1, x_2) and consider the problem of minimizing $p_1'x_1 + p_2'x_2$ subject to

$$A_1x_1 + A_2x_2 \geq b \quad (4.6)$$

$$x_1 \geq 0, \quad (4.7)$$

where p and A have been similarly partitioned. Note that x_2 is not required to be non-negative. These constraints can be put into the form of (4.1) and (4.2) by the substitution

$$x_2 = z - y,$$

where $z \geq 0, y \geq 0$. All variables x_1, z and y are now non-negative.

The dual problem is then, from (4.3) and (4.4),

$$\text{Maximize } b'v \quad (4.8)$$

subject to

$$A_1'v \leq p_1 \quad (4.9)$$

$$A_2'v = p_2 \quad (4.10)$$

$$v \geq 0. \quad (4.11)$$

5. A class of convex programs and its dual

Consider the following class of nonlinear programming problems:

$$\text{Minimize: } f(x) \quad (5.1)$$

subject to

$$Ax \geq b \quad (5.2)$$

$$x \geq 0, \quad (5.3)$$

where $f(x)$ is a convex scalar function of the vector x and possesses continuous first partial derivatives with respect to each component of x . The vectors p, b , and x and the matrix A are the same as in Section 3. Denote this as *Problem I*. Let R_I be the region defined in Euclidian n -space, E_n , by the vectors x satisfying the constraints (5.2) and (5.3). The function $f(x)$ is required to be convex in R_I .

A dual problem to Problem I is

$$\text{Maximize } g(u, v) = f(u) - u' \nabla f(u) + b'v \quad (5.4)$$

subject to

$$A'v - \nabla f(u) \leq 0 \quad (5.5)$$

$$v \geq 0. \quad (5.6)$$

Denote this as *Problem II*. Let R_{II} be the region defined in E_n by the vectors u satisfying the constraints (5.5) and (5.6). The function $f(u)$ will be required to be convex in R_{II} . Problems I and II are related through the theorem in the next section.

6. A duality theorem for convex programs

• Theorem⁴

(i) If there exists a vector x_0 which minimizes $f(x)$ in Problem I, then there also exist vectors $u = x_0, v = v_0$ which maximize $g(u, v)$ in Problem II.

(ii) Conversely, if u_0, v_0 are vectors which maximize $g(u, v)$ in Problem II, then $x = u_0$ minimizes $f(x)$ in Problem I. In either case

$$\text{Minimum } f(x) = \text{Maximum } g(u, v).$$

• Proof

(i) Let x_0 be a solution to Problem I, then

$$f(x_0) \leq f(x). \quad (6.1)$$

Consider the following linear programming problem:

$$\text{Minimize } F(x) = -f(x_0) + x' \nabla f(x_0) \quad (6.2)$$

$$Ax \geq b \quad (6.3)$$

$$x \geq 0. \quad (6.4)$$

Call this *Problem I'*. Note that x_0 is a feasible solution to Problem I' in the sense that it satisfies the constraints (6.3) and (6.4). Suppose there exists another feasible solution x^* to Problem I' with the property that

$$F(x^*) < F(x_0),$$

i.e.,

$$(x^* - x_0)' \nabla f(x_0) < 0. \quad (6.5)$$

It is easily verified that

$$x_1 = x_0 + k(x^* - x_0), 0 \leq k \leq 1$$

is also a feasible solution to both Problem I and Problem I'.

From the mean value theorem

$$\begin{aligned} f(x_1) - f(x_0) &= (x_1 - x_0)' \{ \nabla f[x_0 - \theta(x_1 - x_0)] \} \quad 0 \leq \theta \leq 1 \\ &= k(x^* - x_0)' \{ \nabla f[x_0 - \theta k(x^* - x_0)] \} \\ &= k \{ (x^* - x_0)' \{ \nabla f[x_0 - \theta k(x^* - x_0)] - \nabla f(x_0) \} \\ &\quad + (x^* - x_0)' [\nabla f(x_0)] \}. \end{aligned}$$

The last term in the bold-faced brackets is negative by virtue of the assumption (6.5). Moreover, this term is independent of k . However, since $\nabla f(x)$ is continuous, for k sufficiently small the first term in the bold-faced brackets may be arbitrarily small. In particular k may be chosen so that

$$\begin{aligned} &| (x^* - x_0)' \{ \nabla f[x_0 - \theta k(x^* - x_0)] - \nabla f(x_0) \} | \\ &< | (x^* - x_0)' [\nabla f(x_0)] |. \end{aligned}$$

It follows then from the inequality (6.5) that

$$f(x_1) - f(x_0) < 0.$$

But this is in contradiction to (6.1). Therefore, x_0 must be the minimizing solution of Problem I'. This argument is essentially that given by Kuhn and Tucker⁵ to construct a linear problem whose solution coincides with that of a convex problem. From the duality theorem of linear programming a solution also exists to the dual to Problem I'. This dual linear problem is from (4.3) and (4.4):

$$\text{Maximize } G(v) = -f(x_0) + b'v \quad (6.6)$$

$$A'v \leq \nabla f(x_0) \quad (6.7)$$

$$v \geq 0. \quad (6.8)$$

Call this *Problem II'*. Denote the vector v which maximizes Problem II' by v_0 . Then the duality theorem of linear programming, (4.5), requires that

$$x_0' \nabla f(x_0) = b'v_0. \quad (6.9)$$

Notice now that $u = x_0$, $v = v_0$ comprises a feasible solution to Problem II. To show that it also provides a maximal solution consider

$$\begin{aligned} g(x_0, v_0) - g(u, v) &= f(x_0) - x_0' \nabla f(x_0) + b'v_0 \\ &\quad - f(u) + u' \nabla f(u) - b'v. \end{aligned}$$

From (6.9)

$$g(x_0, v_0) - g(u, v) = f(x_0) - f(u) + u' \nabla f(u) - b'v,$$

and from the lemma, since $f(u)$ is convex in R_{II} ,

$$\begin{aligned} g(x_0, v_0) - g(u, v) &\geq (x_0 - u)' \nabla f(u) + u' \nabla f(u) - b'v \\ &= x_0' [\nabla f(u)] - b'v. \end{aligned} \quad (6.10)$$

Now from (5.2) and (5.6)

$$-b'v \geq -x_0' A'v$$

and from (5.3) and (5.5)

$$x_0' \nabla f(u) \geq x_0' A'v.$$

Substituting these last two inequalities in (6.10),

$$g(x_0, v_0) - g(u, v) \geq x_0' A'v - x_0' A'v = 0.$$

Thus (x_0, v_0) is a maximizing solution for Problem II. Finally from (5.4) and (6.9)

$$g(x_0, v_0) = f(x_0) - x_0' \nabla f(x_0) + b'v_0 = f(x_0),$$

which verifies the equality of the objective functions. This completes the proof of part (i).

The proof of the converse, part (ii), does not follow directly from the above proof since the constraints in Problem II are nonlinear. The proof, however, follows the same general outline as the proof of part (i).

Now recall that R_{II} denotes the range of values of u in Euclidian n -space, E_n , for which

$$A'v - \nabla f(u) \leq 0$$

$$v \geq 0.$$

Let D denote the domain of values of y in E_n for which

$$A'v - y \leq 0.$$

$$v \geq 0.$$

It is clear that R_{II} is mapped onto D by

$$y = \nabla f(u),$$

and that given a point y in D there exists a point u (not necessarily unique) in R_{II} satisfying the last equation above. That is to say, there exists a function which maps D onto R_{II} . It will be assumed that this function is once differentiable,⁶ and moreover that in the neighborhood of the maximum the mapping is one-to-one.

Now let (u_0, v_0, y_0) be a maximizing solution of Problem II rephrased as:

$$\text{Maximize } \psi(u, v, y) = f(u) - u'y + b'v \quad (6.11)$$

$$A'v - y \leq 0 \quad (6.12)$$

$$y = \nabla f(u) \quad (6.13)$$

$$v \geq 0. \quad (6.14)$$

Thus

$$\psi(u_0, v_0, y_0) \geq \psi(u, v, y) \quad (6.15)$$

for all u, v, y satisfying the constraints.

Consider the related linear problem:

$$\text{Maximize } G(v, y) = f(u_0) - u_0'y + b'v$$

$$A'v - y \leq 0$$

$$v \geq 0.$$

Denote this as *Problem II**.

Notice that $v = v_0$, $y = y_0$ is a feasible solution to Problem II*. Suppose there exists v^* , y^* satisfying (6.12) and (6.14) such that

$$G(v^*, y^*) > G(v_o, y_o),$$

i.e.,

$$-u_o'(y^* - y_o) + b'(v^* - v_o) > 0. \quad (6.16)$$

Define vectors v_1, y_1 to be

$$v_1 = v_o + k(v^* - v_o) \quad (6.17)$$

$$y_1 = y_o + k(y^* - y_o) \quad (6.18)$$

for $0 < k \leq 1$. Then v_1, y_1 is also a feasible solution to Problem II*. Since y_1 is in D there exists a vector u_1 in R_{II} such that

$$y_1 = \nabla f(u_1), \quad (6.19)$$

and thus u_1, v_1, y_1 is a feasible solution to Problem II. Consider

$$\begin{aligned} \psi(u_1, v_1, y_1) - \psi(u_o, v_o, y_o) \\ = f(u_1) - u_1' y_1 + b' v_1 - f(u_o) + u_o' y_o - b' v_o. \end{aligned}$$

From the mean value theorem

$$\begin{aligned} \psi(u_1, v_1, y_1) - \psi(u_o, v_o, y_o) \\ = (u_1 - u_o)' \nabla f[u_1 - \theta(u_1 - u_o)] \\ - u_1' y_1 + u_o' y_o + b'(v_1 - v_o) \quad 0 \leq \theta \leq 1 \end{aligned}$$

and using Eq. (6.19)

$$\begin{aligned} \psi(u_1, v_1, y_1) - \psi(u_o, v_o, y_o) \\ = (u_1 - u_o)' \{ \nabla f[u_1 - \theta(u_1 - u_o)] \\ - \nabla f(u_1) \} - u_o'(y_1 - y_o) + b'(v_1 - v_o). \quad (6.20) \end{aligned}$$

Since by assumption u possesses first derivatives with respect to y , where $y = \nabla f(u)$, it follows from the mean value theorem that

$$(u_1 - u_o)' = (y_1 - y_o)' M(y_o - \eta[y_1 - y_o]), \quad 0 \leq \eta \leq 1,$$

where $M(y)$ is a matrix whose i, j element is the partial derivative of the j th component of u with respect to the i th component of y . Using this and Eqs. (6.17) and (6.18), Eq. (6.20) becomes

$$\begin{aligned} \psi(u_1, v_1, y_1) - \psi(u_o, v_o, y_o) = k \{ (y^* - y_o)' M[y_o - \eta k(y^* - y_o)] \\ \{ \nabla f[u_1 - \theta k(u^* - u_o)] - \nabla f(u_1) \} \\ - u_o'(y^* - y_o) + b'(v^* - v_o) \}. \quad (6.21) \end{aligned}$$

Now if k is sufficiently small, the following inequality can be satisfied

$$\begin{aligned} |(y^* - y_o)' M[y_o - \eta k(y^* - y_o)] \{ \nabla f[u_1 - \theta k(u^* - u_o)] \\ - \nabla f(u_1) \}| < |-u_o'(y^* - y_o) + b'(v^* - v_o)|, \end{aligned}$$

since $\nabla f(u)$ is continuous and the term on the right is independent of k .

From the assumption (6.16), the term inside the absolute value signs on the right of the above inequality is positive. Thus the term in bold-faced brackets in Eq. (6.21) may be made positive by a sufficiently small

choice of k . This implies that

$$\psi(u_1, v_1, y_1) - \psi(u_o, v_o, y_o) > 0$$

and contradicts the hypothesis (6.15) that u_o, v_o, y_o is a maximizing solution of Problem II. It follows that v_o, y_o maximizes Problem II*.

Problem II* is a linear problem of the second type mentioned in Section 4, Equations (4.6) and (4.7), and its dual problem according to (4.8) to (4.11) is:

$$\text{Minimize } F(x) = f(u_o) \quad (6.22)$$

$$Ax \geq b \quad (6.23)$$

$$x = u_o \quad (6.24)$$

$$x \geq 0. \quad (6.25)$$

This will be referred to as *Problem I**. The duality theorem of linear programming assures the existence of a solution to this problem. By virtue of (6.24) u_o must be that solution. The duality theorem, (4.5), then provides that:

$$\text{Maximum } G(v, y) = \text{Minimum } F(x)$$

or

$$-u_o' y_o + b' v_o = 0. \quad (6.26)$$

Now u_o , since it satisfies (6.23) and (6.25), is a feasible solution to Problem I. Let x be any other feasible solution to Problem I. From the lemma again, since $f(x)$ is convex in R_I ,

$$f(x) - f(u_o) \geq (x - u_o)' [\nabla f(u_o)]. \quad (6.27)$$

Now from (6.13) and (6.26)

$$-u_o' [\nabla f(u_o)] = -u_o' y_o = -b' v_o,$$

and from (5.2) and (5.6)

$$-b' v_o \geq -x' A' v_o,$$

so

$$-u_o' [\nabla f(u_o)] \geq -x' A' v_o. \quad (6.28)$$

From (5.3) and (5.5)

$$x' [\nabla f(u_o)] \geq x' A' v_o. \quad (6.29)$$

Substituting (6.28) and (6.29) into the right-hand member of (6.27) it follows that

$$f(x) - f(u_o) \geq x' A' v_o - x' A' v_o = 0.$$

Thus u_o minimizes Problem I.

From (6.26) and (5.4)

$$f(u_o) = f(u_o) - u_o' \nabla f(u_o) + b' v_o = g(u_o, v_o).$$

Notice that Problem II may have the constraints⁷

$$u \geq 0 \quad (6.30)$$

added to it without altering any of the foregoing arguments, since this constraint is always satisfied at the optimum.

7. Remarks

It has already been pointed out that the theorem presented here is similar to one previously given by Dennis.² There are, however, some basic differences in the theorems.

Dennis has cast the dual problem (Problem II) in terms of variables y_C and y_L which are related to the variables u and v used here through

$$y_C = \nabla f(u)$$

$$y_L = v.$$

The objective function g then is written in terms of the Legendre transformation of the function $f(u)$. Dennis has shown that the objective function g is a concave⁸ function of the variables y_C , y_L provided f is a strictly convex function, i.e., strict inequality holds in (3.1). This allows the use of the Kuhn-Tucker theorem⁵ to complete the proof of both parts of the theorem.

In general, $g(u, v)$ as defined in (5.4) is not a concave function of u and v . As an example consider $f(x) = -\sin x$ where $x = \pi - \epsilon$ for $\epsilon > 0$ and sufficiently small. Then $f''(\pi - \epsilon) > 0$ and f is strictly convex. But $g(u, v) = \sin u - u \cos u + b'v$ and $\partial^2 g / \partial u^2 = -u \cos u - \sin u$ so for $u = \pi - \epsilon$, $\partial^2 g / \partial u^2 = \pi - 2\epsilon > 0$. Thus g is not concave. The Kuhn-Tucker theorem cannot be utilized, therefore, in the second part of the theorem given here.

The function $f(x)$ in the present theorem is *not* required to be strictly convex. The first part of the theorem is proved under the weaker assumption that $f(x)$ is convex with no additional requirements. To prove the second part of the theorem requires in addition that the inverse gradient of the function f has a derivative, i.e., the matrix M consisting of the second partial derivatives of u with respect to y exists.⁹ If $f(x)$ is assumed to possess continuous second derivatives then

$$f(x) = f(x^*) + (x - x^*)' \nabla f(x^*) + (x - x^*)' M^{-1}(\xi) (x - x^*),$$

where

$$\xi = x^* + \theta(x - x^*), \quad 0 \leq \theta \leq 1.$$

From the lemma, then

$$(x - x^*)' M^{-1}(\xi) (x - x^*) \geq 0,$$

and strict inequality holds if f is strictly convex. Strict convexity, therefore, implies that M^{-1} is positive, definite and thus that its inverse M exists.

The converse is not true as may be seen from the following example.

$$\text{Minimize } f(x) = \begin{cases} -x^3 + 3x^2 - 2x + 1, & x \leq 1 \\ x, & x \geq 1 \end{cases}$$

subject to $x \geq 0$.

The function f is convex but not strictly convex and takes on its minimum value, $1 - 2\sqrt{3}/9$, at $x = 1 - \sqrt{3}/3$. The inverse of the relationship $y = f'(x)$ possesses a de-

rivative with respect to y everywhere. The dual problem can be written

$$\text{Maximize } g(u) = \begin{cases} 2u^3 - 3u^2 + 1, & u \leq 1 \\ 0, & u \geq 1 \end{cases}$$

subject to

$$u \geq 1 - \sqrt{3}/3.$$

The function g is neither convex nor concave but takes on its maximum value, $1 - 2\sqrt{3}/9$, at $u = 1 - \sqrt{3}/3$ as was to be expected.

The assumption that M exists is then also weaker than Dennis' assumption of strict convexity.

Finally Dennis has not specified the regions in n -space over which the function f should be required to be convex. The proof here is valid when $f(x)$ is convex over the regions R_I defined by (5.2) and (5.3) and R_{II} defined by (5.5) and (5.6).

It is clear that f must be convex over R_I for otherwise a solution to Problem I may not exist. That it is also necessary that f be convex over R_{II} may be seen by considering the following example.

$$\text{Minimize } f(x_1, x_2) = -\sin x_1 - \sin x_2$$

where

$$-x_1 - x_2 \geq -\pi/3$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

The function f is convex over the domain defined by the constraints and takes on its minimum value, -1 , at $x_1 = x_2 = \pi/6$. The dual problem is:

$$\text{Maximize } g = -\sin u_1 - \sin u_2 + u_1 \cos u_1 + u_2 \cos u_2 - \frac{\pi}{3} v,$$

where

$$-v + \cos u_1 \leq 0 \quad (7.1)$$

$$-v + \cos u_2 \leq 0 \quad (7.2)$$

$$v \geq 0. \quad (7.3)$$

Now notice that $u_1 = u_2 = 2k\pi + \pi/6$ and $v = \sqrt{3}/2$ comprise a feasible solution to the dual problem for any $k = 1, 2, 3, \dots$. But for these values of the variables

$$g = 2\sqrt{3}k\pi - 1,$$

which is unbounded as k becomes large. Thus the solution of the dual problem does not exist. The difficulty is, of course, that $f(u_1, u_2)$ is not convex over the region R_{II} defined by (7.1), (7.2), (7.3) above.

The proof of Section 6 provides a method for computing the dual variables v_o once the primal variables x_o have been obtained. Indeed, v_o is the solution of the following linear programming problem:

$$\text{Maximize } b'v$$

subject to

$$A'v \leq \nabla f(x_0)$$

$$v \geq 0.$$

The constraints of the primal problem (Problem I) may be formulated in other ways with corresponding changes in the dual constraints. Some of these are tabulated below. In all cases the functionals to be optimized are the same as those in Section 5.

Type	Problem I	Problem II
I	$Ax \geq b$ $x \geq 0$	$A'v - \nabla f(u) \leq 0$ $v \geq 0$
II	$Ax \geq b$	$A'v - \nabla f(u) = 0$ $v \geq 0$
III	$Ax = b$ $x \geq 0$	$A'v - \nabla f(u) \leq 0$
IV	$Ax = b$	$A'v - \nabla f(u) = 0$

8. Example

As a simple example of the theorem of Section 6, consider the problem of minimizing

$$f(x_1, x_2) = -\log x_1 - \log x_2 \quad (8.1)$$

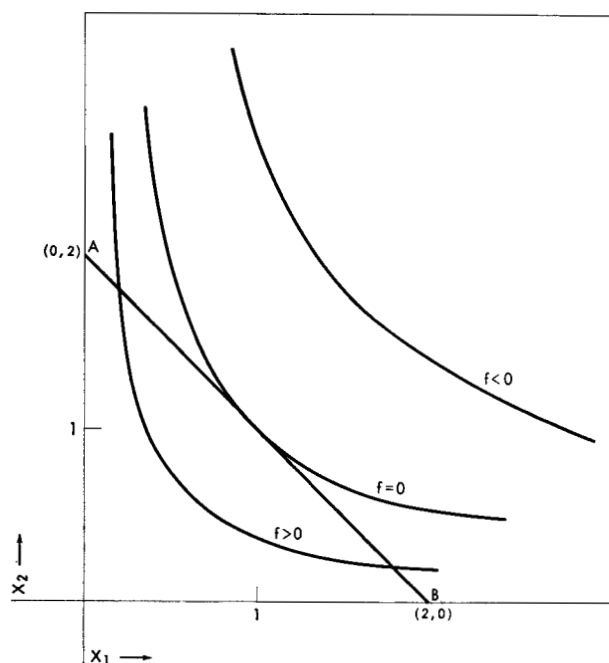
subject to

$$-x_1 - x_2 \geq -2 \quad (8.2)$$

$$x_1 \geq 0 \quad (8.3)$$

$$x_2 \geq 0. \quad (8.4)$$

Figure 1 Example of convex program.



The admissible values of x_1, x_2 lie in the triangle OAB of Fig. 1. The contours of constant values of f are also shown with f decreasing as the contours recede from the origin.

The optimal value is $f=0$ for $x_1=x_2=1$.

The dual problem is:

$$\text{Maximize } g(u_1, u_2, v) = -\log u_1 - \log u_2 + 2 - 2v, \quad (8.5)$$

where

$$-v + \frac{1}{u_1} \leq 0 \quad (8.6)$$

$$-v + \frac{1}{u_2} \leq 0 \quad (8.7)$$

$$v \geq 0$$

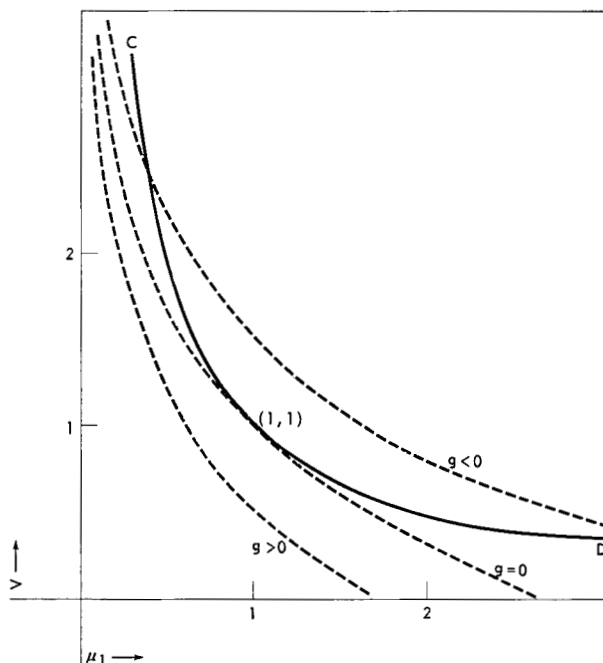
$$u_1 \geq 0 \quad (8.8)$$

$$u_2 \geq 0.$$

The last two inequalities are added in order that $g(u_1, u_2, v)$ may be defined over the constraint set. The addition of these constraints is permissible in view of the remark at the close of Section 6. (See Eq. (6.30).)

From the symmetry of the dual it is clear that when g is a maximum, $u_1 = u_2$. The admissible values of u_1 and v are those above the hyperbola CD in Fig. 2. The contours of constant g are shown as dotted curves with g increasing as the contours approach the origin. The maximum value of g is associated with the contour which is tangent to the hyperbola CD at $u_1 = u_2 = v = 1$. For these values of the variables $g(1, 1, 1) = 0$.

Figure 2 Example of dual convex program.



Note that $f(x_1, x_2)$ is convex in the regions R_I and R_{II} defined by the constraints (8.2), (8.3), (8.4) and (8.6), (8.7), (8.8).

9. Acknowledgment

The author is indebted to Professor C. E. Lemke of Rensselaer Polytechnic Institute for his comments and suggestions.

References and footnotes

1. W. S. Dorn, "Duality in Quadratic Programming," *Quarterly of Applied Mathematics*, **18**, 155-162 (July, 1960).
2. Jack B. Dennis, *Mathematical Programming and Electrical Networks*, Technology Press and John Wiley & Sons, New York, N. Y., 1959.
3. D. Gale, H. W. Kuhn, and A. W. Tucker, "Linear Programming and the Theory of Games," Chapter XIX, *Activity Anal. of Prod. and Alloc.*, Cowles Comm. Mono. 13, John Wiley & Sons, New York, N. Y., 1951.
4. Dennis² has stated and proved a duality theorem which is similar to but less general than this theorem. The relationship between the theorems will be discussed in Section 7.
5. H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," *Proc. 2nd Berkeley Symposium in Math. Stat. and Prob.*, 481-492 (August, 1950), University of California Press, 1951.
6. Note that this assumption was not required for the proof of the first part of the theorem.
7. The constraints $Au \geq b$ are also satisfied by the optimum vector u .
8. That is to say, its negative is convex.
9. See Section 6 immediately following Eq. (6.20).

Received October 28, 1959