

On the Switching Time of Subharmonic Oscillators

Abstract: The time necessary to change the phase of an idealized subharmonic parametric oscillator from one value to a value differing by 180° is calculated for various values of pump power and switching power. Several conclusions are drawn from the results.

There exists at present a great interest in variable-reactance devices. One of the possible uses of such variable reactances is as parametric subharmonic oscillators, which have potential applications in computer technology.^{1,2} These applications depend upon the fact that the subharmonic oscillations can occur at one of two phases differing by 180° . The value of the phase (0° or 180°) can form the basis of a binary arithmetical system.

In such an application, three of the important properties of the subharmonic oscillations are the time it would take to switch the oscillator from one phase to the other, the pump power necessary for satisfactory oscillation, and the efficiency of subharmonic production. Since these properties all depend on the same parameters, it is not possible to optimize each one individually. This paper will discuss the switching time of the subharmonic oscillator and will then briefly discuss the interrelationships between this time and the pump power and efficiency.

The circuit to be analyzed is shown in Fig. 1 and is similar to that given by Bloom and Chang.^{3,4} The approximations used will also be similar but may be somewhat less applicable in a real situation, since the level of the pump will be greater for the subharmonic oscillator than for the parametric amplifier. Here L is the variable reactor,⁵ where the flux ϕ is given by

$$\phi = L_0 I - \mathcal{L} I^2. \quad (1)$$

This equation does not necessarily represent the exact behavior of any real inductor, since it is only the first two terms of a Taylor expansion. However, it is the first approximation to the behavior of a biased core.⁶ The same approximation has been used previously.³ Power from the pump at frequency ω_3 is sent through the circuit composed of C_3 (where C_3 is chosen such that $\omega_3 = 1/\sqrt{C_3 L_0}$) and the variable inductor. This power is in part absorbed by the resistance of the variable inductor, R_L , and in part

converted to the subharmonic frequency, $\omega_1 = \omega_3/2$. A part of this latter power appears as useful output in the load resistor, R_1 . Again, C_1 is chosen such that $\omega_1 = 1/\sqrt{L_0 C_1}$. It will be assumed that the resonant circuits or cavities are lossless, the only resistances involved being those of the variable inductor and of the output circuit. Also, it will be assumed that the circuit Q 's are high enough that the voltages developed at all other frequencies by the nonlinear element are negligible.

The equations connecting the currents and voltages of the two circuits subject to the above assumptions are

$$I_1(R_1 + R_L) - j\omega_1 \mathcal{L} I_3 I_1^* = 0 \quad (2a)$$

$$I_3 R_L - j\omega_3 \mathcal{L} I_1^2 / 2 = V_3, \quad (2b)$$

where ordinary complex notation is used. These equations are found from the relation for the voltage across the reactor

$$V = d\phi/dt. \quad (3)$$

This is most easily shown by writing the current I as

$$I = (I_1 e^{j\omega_1 t} + I_1^* e^{-j\omega_1 t}) / 2 + (I_3 e^{j\omega_3 t} + I_3^* e^{-j\omega_3 t}) / 2,$$

where I_1 and I_3 may be complex to allow for possible

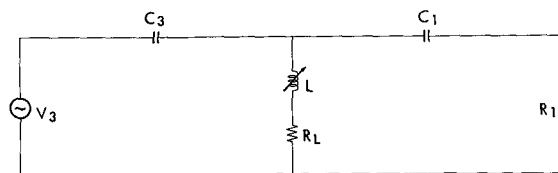


Figure 1 Schematic representation of subharmonic oscillator.

phase differences. This expression is substituted into (1) to get ϕ , which in turn is substituted in (3). Voltage and current terms of the same frequency are then collected and the net potential around both loops is equated to zero. The terms of major interest are given by

$$2d(\mathcal{L}I^2)/dt = -j\omega_1\mathcal{L}(I_1^2e^{j\omega_1t} - I_1^{*2}e^{-j\omega_1t}) \\ - I_1I_3^*e^{-j\omega_1t} + I_1^*I_3e^{j\omega_1t}.$$

In expressions (2) only the positive exponential terms are included. More detail on setting up the equations has been given previously.³ However, this reference does not specifically treat the present case of the subharmonic oscillator, where the pump power is greater than threshold.

From Eq. (2), if we let $R_T = R_1 + R_L$,

$$I_3 = \frac{R_T}{j\omega_1\mathcal{L}} \frac{I_1}{I_1^*} = \frac{1}{jQ_1} \left(\frac{L}{\mathcal{L}} \right) \left(\frac{I_1}{I_1^*} \right) \quad (4)$$

and

$$I_1I_1^* = \frac{R_LR_T}{\omega_1^2\mathcal{L}^2} \left(\frac{V_3j\omega_1\mathcal{L}}{R_LR_T} \frac{I_1^*}{I_1} - 1 \right). \quad (5)$$

Since $I_1I_1^*$ must be real and positive, it is seen that $I_1^*/I_1 = -j$. This can only be true for subharmonic phases of $\pi/4$ and $5\pi/4$, which of course differ by π radians. Thus the undriven subharmonic oscillation occurs at one of two phases which differ by 180° . It is seen that the small reactance terms introduced by the nonlinearity, and which are present for the degenerate parametric amplifier,³ are not present for the steady state oscillations of the subharmonic oscillator (with pump power greater than threshold). It is interesting to note that the pump current I_3 remains constant at the value $R_T/\omega_1\mathcal{L}$ once the threshold pump power for oscillation is exceeded.

If we define α as

$$\alpha = \frac{\omega_1\mathcal{L}V_3}{R_LR_L} = \frac{I_3^*V_3}{I_3I_3^*R_L} = \frac{P_p}{P_c} \quad (6)$$

(where P_p is the pump power and P_c the threshold power for oscillation), the equilibrium value of $I_1I_1^*$ attained is

$$I_1I_1^* = \frac{R_LR_T}{\omega_1^2\mathcal{L}^2} (\alpha - 1) \\ = \frac{R_L}{R_T} \frac{1}{Q_1^2} \left(\frac{L}{\mathcal{L}} \right)^2 (\alpha - 1). \quad (7)$$

Also,

$$P_c = \frac{R_L}{2} \frac{R_T^2}{\omega_1^2\mathcal{L}^2} \\ = \frac{1}{2Q_3Q_1^2} \left(\frac{L_0}{\mathcal{L}} \right)^2 \frac{1}{L_0\omega_3}. \quad (8)$$

For a low threshold power, a high Q is especially important.

The efficiency of conversion of pump power to subharmonic power is

$$\eta = \frac{I_1I_1^*R_1}{V_3I_3^*} \\ = \frac{R_1}{R_T} \cdot \frac{I_1I_1^*}{I_1I_1^* + (R_L/R_T)(1/Q_1^2)(L/\mathcal{L})^2}. \quad (9)$$

Thus, the efficiency increases to a value of R_1/R_T for very large $I_1I_1^*$ (which implies a very large pump power, P_p).

There are several methods by which the subharmonic oscillator could be switched to its state of opposite phase. For example, to reduce the energy storage at the unwanted pre-existing phase, the pump power could instantaneously be reduced or turned off, or the bias could instantaneously be made very positive. Other methods are also possible. The method that will be discussed here is the introduction of a signal of opposite phase at the subharmonic frequency and which is large enough to override the pre-existing subharmonic output. It is assumed that the pump power remains constant at a given level (above threshold). In what follows, we will assume that this switching generator is of the constant-voltage type, so that the power supplied is proportional to the voltage of the generator, V_1 , times the current in the circuit, I_1 . Use of a different switching generator (for example, a constant-power generator) would influence the results somewhat, but the qualitative conclusions would be unchanged.

Equation (7) for the steady state of the subharmonic oscillator can be written in the alternative form

$$\alpha R_T I_1^2 (1 - I_1^2/i_0^2) - R_T I_1^2 = 0, \quad (10)$$

where $i_0^2 = \alpha R_T R_L / \omega_1^2 \mathcal{L}^2 = \alpha P_c / R_T$ and the currents are magnitudes only (which may, however, be plus or minus to denote the two possible values of phase differing by 180°). This expression correctly gives the various steady state properties of the undriven subharmonic oscillator. The first term in Eq. (10) is twice the power produced by the subharmonic action and the second, twice the power lost in Joule heating.

When the transient switching behavior is considered, the power generated by the external switching generator, $\frac{1}{2}V_1I_1$, must be added, as well as the rate of change of stored energy, $L_0I_1dI_1/dt$. It is assumed that V_1 is exactly of the same or opposite phase to I_1 . Thus, the rate of change of stored energy is equal to the balance of the power generated by the subharmonic action, the Joule losses, and the power supplied by the external switching generator:

$$2L_0I_1dI_1/dt = \alpha R_T I_1^2 (1 - I_1^2/i_0^2) - R_T I_1^2 + V_1I_1. \quad (11)$$

It can be noted that Eq. (11) reduces to the expected result for various special cases. For the steady state undriven oscillation ($V_1=0$, $dI_1/dt=0$), it reduces to Eq. (7). It also correctly predicts the time for a small signal to increase by a factor "e". Eq. (11) reads for this case $2L_0dI_1/dt = (\alpha-1)R_TI_1$, or $\ln I_1 = (\alpha-1)R_Tt/2L_0 = (\alpha-1)\omega_1t/2Q_1$. Therefore the exponential build-up time, τ' , is just $(2Q_1/\omega_1) \cdot [1/(\alpha-1)]$, as has been previously

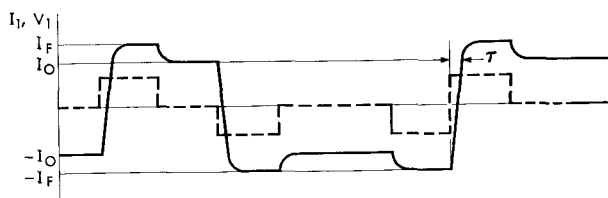


Figure 2 **Switching voltage and subharmonic current versus time.**

A change in the sign of the current corresponds to a phase change of 180° .

shown.⁷ With the saturation term (I_1^2/i_0^2) included, Eq. (11) could be integrated to give this same build-up time with saturation included. Eq. (11) in its steady state form $(2L_0I_1dI_1/dt=0)$ can also be derived directly from Eqs. (2a) and (2b) if V_1 is substituted for the zero in the left side of Eq. (2a). It should be noted that Eq. (11) predicts considerable limiting action, since the power developed by the subharmonic action decreases abruptly when I_1 becomes comparable to i_0 .

The main use to be made here of Eq. (11) will be to calculate the time necessary to change the phase of an established current, I_1 , by 180° by the application of a large enough signal, V_1 , of phase exactly opposite to that of I_1 . The time taken to change the current from an initial value $-I_F$ to a final value I_0 (a change in the sign of I is equivalent to a phase change of 180°) is given from the above by

$$t_0 - t_F = \frac{2L_0}{R_T} \int_{-I_F}^{I_0} \frac{dI_1}{\alpha I_1(1 - I_1^2/i_0^2) - I_1 + V_1/R_T} \\ = \frac{2Q_1}{\omega_1} \int_{-\mathcal{G}_F}^{\mathcal{G}_0} \frac{d\mathcal{G}_1}{\alpha \mathcal{G}_1(1 - \mathcal{G}_1^2) - \mathcal{G}_1 + c}, \quad (12)$$

where $c = V_1/R_T i_0$ and $\mathcal{G}_1 = I_1/i_0$. A similar analysis has been made⁸ for the case of the subharmonic klystron. However, in that case the integral could not be written in analytic form.

The question remains as to what limits should be chosen for the integration. Figure 2 shows the type of operation that might be expected in a computer. The switching signals are shown as the dashed curve and the subharmonic output as the solid curve. The time constant that will be computed is τ , the time it takes the current to change from the state $\pm I_F$ to the state $\mp I_0$, where the initial value is chosen to be the equilibrium current with switch on and the final value that with switch off. This is a somewhat conservative measure which takes a longer time than, for example, $\pm I_0$ to $\mp I_0$. The time would not be shortened very much if a fraction of $\mp I_0$, such as $\mp I_0/2$, were chosen as the final state. The value of the final state is given by $\mathcal{G}_0 = I_0/i_0 = \sqrt{(\alpha - 1)/\alpha}$. The initial state, $-\mathcal{G}_F = -I_F/i_0$, is determined by both the values of α and of c , where c is proportional to the switching voltage V_1 . It is found by solution of the equation $\alpha \mathcal{G}_F(1 - \mathcal{G}_F^2) - \mathcal{G}_F = c$.

The integral in Eq. (12) can either be integrated by parts or can be solved numerically for various values of c and α . The results are expressed in Fig. 3 as the time for switching in units $2Q_1/\omega_1$ versus $\alpha = P_p/P_c$ for various values of $\mathcal{G}_F$, which depends in turn on the amplitude of the switching voltage, V_1 . It is seen that times shorter than the passive decay time ($\tau_0 = 2Q_1/\omega_1$) are obtained

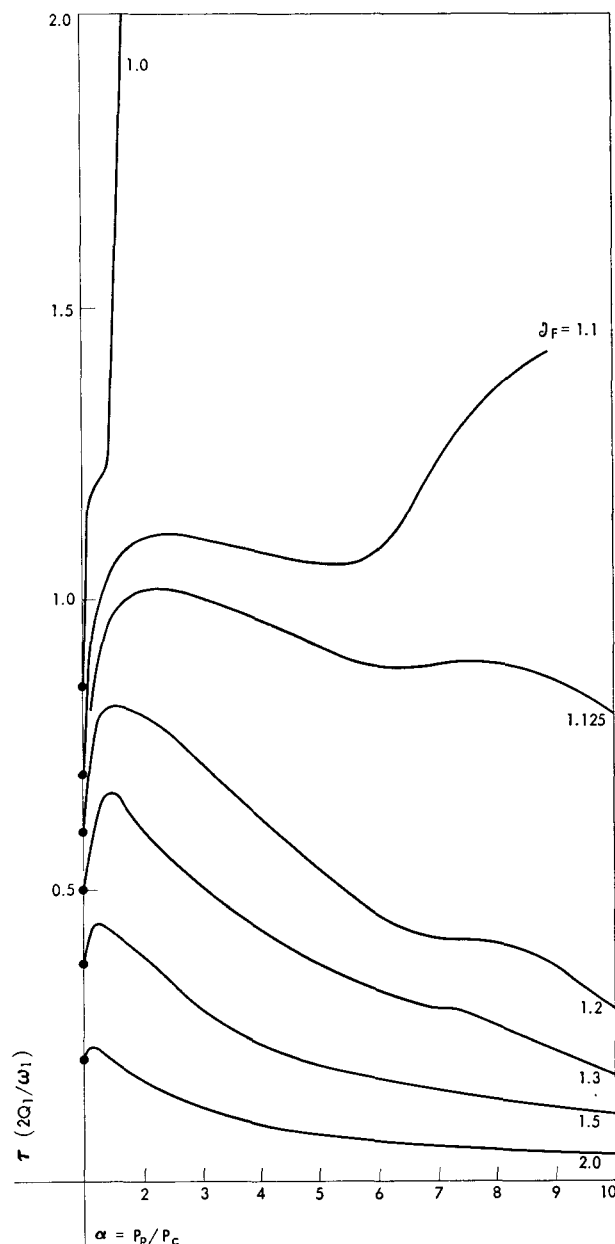


Figure 3 **The switching time of the subharmonic oscillator.**

This time is plotted against the pump power with the parameter being the normalized steady-state subharmonic current when driven by the switching generator. This parameter increases as the switching voltage increases.

if $\mathcal{G}_F > 1.125$, but that the switching time rises very rapidly if the switching power is less than that which corresponds to $\mathcal{G}_F = 1.125$. The results shown in Fig. 3 are, of course, equally applicable for both nonlinear inductors and nonlinear capacitors.

These results can also be plotted with $\mathcal{G}_F/\mathcal{G}_0 = I_F/I_0$ as the switching parameter. This is shown in Fig. 4. It is seen here that, for small values of α , large values of I_F/I_0 are necessary for rapid switching and vice versa. Very roughly, it would appear that the pump power should be at least several times the threshold power in order for the switching to be reasonably fast without excessive switching power.

This is also a requirement for reasonable efficiency of operation, as can be seen from Eq. (9). However, fast switching times and low threshold power are mutually exclusive. The first demands a low Q and the second a high Q . The scale of the ordinates in Fig. 4 is proportional to the Q and, even though the scale of the abscissae is proportional to Q^3 for constant pumping power, the ordinates take precedence except at the left side of the figure. Therefore, the switching time rises for an increase in Q except for small pumping power, where the times are long in any event.

These times can be compared with the time necessary for a small signal to be amplified by a factor "e". This time is given by the expression

$$\tau' = \frac{2Q_1}{\omega_1} \left(\frac{1}{\alpha - 1} \right). \quad (13)$$

This time is shown as the dashed curve in Fig. 4 and is comparable to the switching time for reasonably large switching signals ($I_F/I_0 \approx 1.7$).

It can be shown that it is possible for one subharmonic oscillator to switch another identical subharmonic oscillator, but with an increase in switching time to approximately $10Q_1/\omega_1$ ($\tau \approx 5$ in Fig. 4).⁹ Several simplifications will be made. First it will be assumed that an isolator or some other one-way device prevents power from flowing from Oscillator 2 to Oscillator 1. Second, it will be assumed that power from Oscillator 1 flows only into the reactor of Oscillator 2 and not into its load or into any other subsequent circuitry. There is a certain imprecision in setting up the constant-voltage generator required in Eq. (12). It will be assumed that the voltage, V_1 , induced in Circuit 2 is equal to the current supplied by Circuit 1, I_0 , multiplied by the reactor resistance $R_L = R_T/2$, or $V_1 = I_0 R_T/2$. The power delivered is thus $I_0 I_1 R_T/4$, where I_1 varies from a value $-I_F$ to $+I_0$. Over the range of I_1 from $-I_F$ to $-I_0$, the power delivered is greater than the power available. However, the time spent in traversing the region $-I_F$ to $-I_0$ is only a small part of the total switching time, and in any event the result of such a calculation will set a lower limit on the switching time subject to the constant-voltage assumption.

From the denominator of Eq. (12) it can be seen that, for switching to occur,

$$V_1/R_T = I_0/2 > \alpha I_1(1 - I_1^2/I_0^2) - I_1 \quad (14)$$

for all values of I_1 . The right side of the equation has its maximum value for $\mathcal{G}_1 = I_1/I_0 = \sqrt{(\alpha - 1)/3\alpha}$. Substitution of this value into Eq. (14) leads to the condition that $\alpha = P_p/P_c < 2.25$ for switching to occur. For values of $\alpha < 2.25$, the value of c to be used in Eq. (12) is $c = \mathcal{G}_0/2 = \sqrt{(\alpha - 1)/4\alpha}$. A table of the calculated values of the time in units $2Q_1/\omega_1$ and of I_F/I_0 are given in Table 1 for the possible switching of one oscillator by another.

Although it is possible to switch one oscillator by another, the times are quite long and operation would be in a region where rather poor efficiency in conversion of pump power to subharmonic power would be obtained.

In this paper, the compromises inherent in trying to simultaneously minimize pump power, switching power, and switching time have been considered for the case where a subharmonic oscillator has impressed upon it a

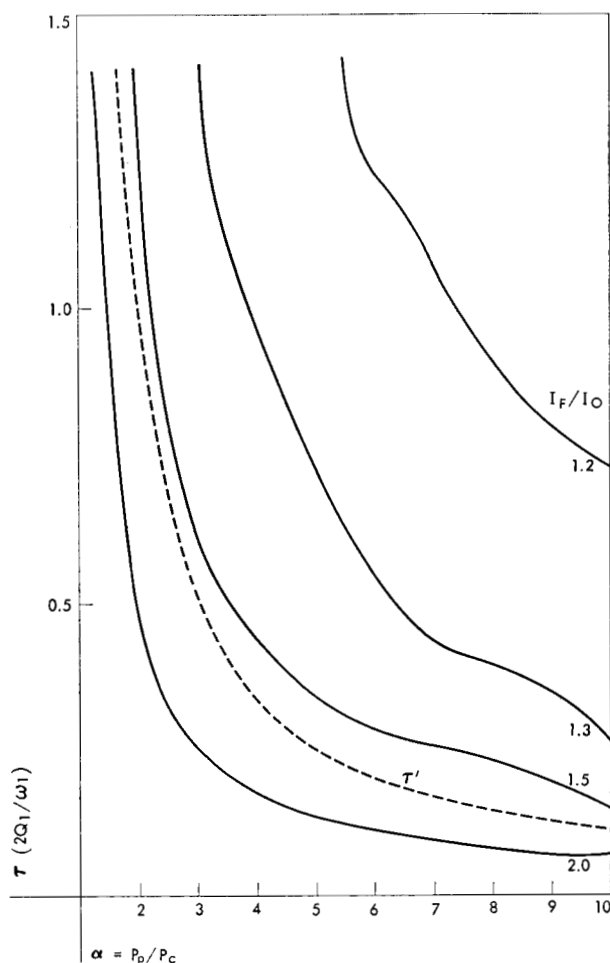


Figure 4 The switching time of the subharmonic oscillator.

This time is plotted against the pump power with the parameter being the ratio of the steady-state subharmonic current when driven by the switching generator to that when not driven.

switching signal of opposite phase. The method followed should also be of use in the treatment of more complicated engineering situations.

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References and footnotes

1. J. von Neumann, "Nonlinear Capacitance or Inductance Switching, Amplifying, and Memory Organs," *U. S. Patent #2,815,488*, December 3, 1957.
2. E. Goto, *Denki Tsushin Gakkai-shi*, **38**, 770 (1955).
3. S. Bloom and K. K. N. Chang, *RCA Review*, **18**, 578 (1957).
4. This figure is somewhat schematic and might be clearer if it is considered that ideal filters are present in both arms to prevent currents of the other frequency from circulating in these arms.
5. A variable capacitor could equally well have been chosen. The present choice was made for historical reasons. However, all the results are equally applicable to a variable capacitor if the obvious changes in notation are made.
6. The next (cubic) term in the Taylor expansion contributes no terms of the correct frequencies to Eq. (2). The following (quartic) term does contribute two terms. These terms are such that they decrease the value of the current

Table 1 Values of switching time and excess current for various values of α for switching of one oscillator by another.

α	τ	I_F/I_0
2.0	6.3	1.18
1.5	4.8	1.33
1.25	5.1	1.53
1.1	5.1	1.83

at which limiting begins (i_0) and also add an additional limiting term increasing as I_1^6 to Eq. (10). This is true if the quartic term has the same sign as the quadratic term, as is usually the case. Therefore, these higher terms in the Taylor expansion would ordinarily still further reduce the switching time because of the more effective limiting.

7. J. Hillibrand and W. R. Beam, *RCA Review*, **20**, 229 (1959).
8. N. M. Kroll and I. Palocz, *IBM Journal*, **3**, 345 (1959).
9. That this possibility exists was first shown in Reference 7.

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