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On Dimensional Analysis

Abstract: The dimensions of physical quantities q are interpreted as vectors

$$q_i(\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}) \equiv \gamma_{i1}b_1 + \gamma_{i2}b_2 + \ldots + \gamma_{in}b_n ,$$

where the basic elements b_j generating the vector space represent the basic quantities of the dimensional system and the coefficients γ_j are defined by

$$q_i = \prod_{j=1}^{n} b_j^{\gamma_{ij}} .$$

This interpretation permits the application of the theorems on vector spaces to dimensional analysis. Some results of this approach are simplified rules for the transformation of dimension and unit systems and a physically more transparent derivation of a complete set of dimensionless products by a transformation of bases. The new notation yields a sequential order of physical equations which may lead to a dimensional analysis based on appropriately selected equation groups.

Introduction

For many years dimensional analysis has been a well established field.[1–4] It is mainly used in checking the correctness of physical equations, transforming units, and analyzing similarity conditions. It can also be of aid in deriving physical formulas.

Dimensional analysis is based physically on the concept of homogeneity* and on Buckingham's famous Π-theorem,[5, 6] and mathematically on the theory of simultaneous linear equations. It shall be shown that a slightly different physical approach leads to a formulation which is equivalent to the content of the Π-theorem but is physically more transparent. In many cases, the resulting calculations will be simplified.

Dimension space

A physical formula expresses the functional relation between measurable quantities such as mass, energy, magnetic induction, et cetera. Dimensionally these quantities q_i are defined as products of powers of a restricted number n of basic quantities b_j:

$$q_i = \prod_{j=1}^{n} b_j^{\gamma_{ij}} . \tag{1}$$

The product $\prod b_j^{\gamma_{ij}}$ is called the dimension of the quantity q. It is possible to choose different sets b_j, b'_j, b''_j, . . . of basic quantities defining different dimensional systems. Examples for such systems are mass, length, time (MLT), or electric charge, mass, length, time ($QMLT$).

*Bridgman uses the term *complete* instead of *homogeneous.*

It should be mentioned that a one-to-one correspondence between quantities and dimensions does not always exist. In MLT, for instance, energy and torque are both represented by the dimension ML^2T^{-2}. This introduces a certain arbitrariness into dimensional analysis, the consequences of which shall not be discussed here.

Instead of using the notation of Eq. (1), one may represent the physical quantities q_i in a given dimensional system by vectors[7]

$$\begin{aligned} q_i(\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}) &\equiv [\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}] \\ &\equiv \gamma_{i1}b_1 + \gamma_{i2}b_2 + \ldots + \gamma_{in}b_n . \end{aligned} \tag{2}$$

These vectors can be interpreted as radius vectors in an n-dimensional vector space generated by the set of n linearly independent elements $b_j (i=1, 2, \ldots, n)$. Any such n-dimensional vector space shall be called a dimension space (DS). According to this interpretation, the theorems on vector spaces may be applied to dimensional analysis.

The transformation of bases

Dimensional systems with various numbers n of basic elements have been used in physics. Two vector spaces V_n and V'_m, however, are isomorphic only for $m=n$. A one-to-one correspondence $q_i(\gamma_{ij}) \leftrightarrow q_i(\gamma'_{ij})$ and consequently invariance of physical formulas with respect to a transformation from one dimensional system into another, can thus be expected only if both systems are generated by an equal number n of basic quantities and if these quantities span DS_n.

On the other hand, any n-dimensional vector space V_n is isomorphic with any other n-dimensional vector space.

The transform T of the row matrices $[\gamma'_j]_i \rightarrow [\gamma_j]_i$ is found easily from the system of linear equations:

$$\begin{aligned} b'_1 &= \gamma_{11}b_1+\gamma_{12}b_2+\ldots+\gamma_{1n}b_n \\ b'_2 &= \gamma_{21}b_1+\gamma_{22}b_2+\ldots+\gamma_{2n}b_n \\ &\vdots \\ b'_n &= \gamma_{n1}b_1+\gamma_{n2}b_2+\ldots+\gamma_{nn}b_n \end{aligned} \tag{3}$$

as the matrix $[\gamma_{ik}]$:

$$T=\begin{bmatrix} \gamma_{11}\gamma_{12}\ldots\gamma_{1n} \\ \gamma_{21}\gamma_{22} \\ \vdots \\ \gamma_{n1}\gamma_{n2}\ldots\gamma_{nn} \end{bmatrix} . \tag{4}$$

It can be shown that the transformation of quantities is given by the equation:

$$[\gamma_j]=[\gamma'_j]T \tag{5}$$

In Eq. (5), the vectors $[\gamma_j]$ and $[\gamma'_j]$ may be replaced by matrices of m such vectors:

$$[\gamma_{ij}]=[\gamma'_{ij}]T, \quad i=1,\ 2,\ldots,m\ . \tag{6}$$

It is often desirable to write explicitly the system of linear equations transforming the individual coefficients γ_j of any quantity:

$$\begin{aligned} \gamma_1 &= a_{11}\gamma'_1+a_{12}\gamma'_2+\ \ldots\ +a_{1n}\gamma'_n \\ \gamma_2 &= a_{21}\gamma'_1+a_{22}\gamma'_2+\ \ldots\ +a_{2n}\gamma'_n \\ &\vdots \\ \gamma_n &= a_{n1}\gamma'_1+a_{n2}\gamma'_2+\ \ldots\ +a_{nn}\gamma'_n \end{aligned} \tag{7}$$

The matrix $[a_{ik}]$ is given by

$$[a_{ik}]=T'\ , \tag{8}$$

where T' is the transpose of T.

Transformations between the commonly used dimensional systems

Two mechanical systems are frequently used: mass, length, time (MLT) and force, length, time $(FLT)'$. The equations transforming the coefficients γ_j are given by:

$$\begin{aligned} \gamma_M &= \gamma'_F & \gamma'_F &= \gamma_M \\ \gamma_L &= \gamma'_F+\gamma'_L & \gamma'_L &= -\gamma_M+\gamma_L \\ \gamma_T &= -2\gamma'_F+\gamma'_T & \gamma'_T &= 2\gamma_M+\gamma_T\ . \end{aligned} \tag{9}$$

Energy, written conventionally ML^2T^{-2} transforms thus as follows:

$$\{1M+2L-2T\}\rightarrow\{1F+(-1+2)L+(2-2)T\}=\{1F+1L\}\ .$$

The result corresponds to the conventional notation FLT°. Equation (9) transforms all mechanical quantities. Three basic quantities generate, therefore, the physical subspace of mechanics.

The equations of transformation between the dimensional system $QMLT$ and electric charge, electric potential, length, time ($QVLT$) are:

$$\begin{aligned} \gamma_Q &= \gamma'_Q-\gamma'_V & \gamma'_Q &= \gamma_Q+\gamma_M \\ \gamma_M &= \gamma'_V & \gamma'_V &= \gamma_M \\ \gamma_L &= 2\gamma'_V+\gamma'_L & \gamma'_L &= -2\gamma_M+\gamma_L \\ \gamma_T &= -\gamma'_Q-2\gamma'_V+\gamma'_T & \gamma'_T &= \gamma_Q+3\gamma_M+\gamma_T\ . \end{aligned} \tag{10}$$

The electromagnetic-mechanical subspace is spanned by $n=4$ linearly independent quantities. Consequently, this DS cannot be generated by only 3 basic vectors. It is thus not surprising that no unique transformation exists between the electromagnetic system $(MLT)_{emu}$ and the electrostatic system $(MLT)_{esu}$.

Unique transformations exist, however, from any 4-dimensional electromagnetic system such as $QMLT$ to $(MLT)_{emu}$ and $(MLT)_{esu}$. For instance:

$$\begin{aligned} \gamma_M(esu) &= \tfrac{1}{2}\gamma_Q+\gamma_M & \gamma_M(emu) &= \tfrac{1}{2}\gamma_Q+\gamma_M \\ \gamma_L(esu) &= \tfrac{3}{2}\gamma_Q+\gamma_L & \gamma_L(emu) &= \tfrac{1}{2}\gamma_Q+\gamma_L \\ \gamma_T(esu) &= -\gamma_Q+\gamma_T & \gamma_T(emu) &= \gamma_T\ . \end{aligned} \tag{11}$$

These transformations can be regarded as projections of DS_4 onto the 3-dimensional mechanical subspace. No unique inverse transform exists.

The inclusion of the thermodynamic subspace requires 5 basic quantities to span DS. An appropriate dimensional system is electric charge, mass, length, time, temperature ($QMLT,\theta$). It is compatible with the MKS unit system† and isomorphic with any other 5-dimensional DS. All other dimensional systems in use can be derived from $QMLT,\theta$ by a unique transformation whereas the inverse is not the case.

Tables 1 and 2 give a list of the dimensions and units of the common physical quantities in $QMLT,\theta$. The concept of dimensional vectors permits an extremely condensed notation of physical dimensions as long as no transformations between different dimensional systems are involved. A few quantities may be given as examples

Quantity	*Conventional Notation*	*New Notation in QMLT,θ*
Acceleration	$M^0L^1T^{-2}$	$[1\ \underline{2}]$
Energy	$M^1L^2T^{-2}$	$[1\ 2\ \underline{2}]$
Entropy	$M^1L^2T^{-2}\theta^{-1}$	$[1\ 2\ \underline{2},\ \underline{1}]$
Electric Potential	$Q^{-1}M^1L^2T^{-2}$	$[\underline{1}\ 1\ 2\ \underline{2}]$

Isomorphism of two dimensional systems is the necessary and sufficient condition for invariance of homogeneous physical formulas with regard to a transformation from one system into the other. Invariance means that the form of a formula does not depend on the choice of a particular basis of DS.

The conversion of units

A definite physical quantity q is given by its measure m and its unit $u(q)$:

$$q\equiv mu(q)\ . \tag{12}$$

In another unit system, q may be written

$$q\equiv m'u'(q)\ . \tag{13}$$

†The MKS-system with the units meter, kilogram, second, ampere is legally accepted in the United States since January, 1948.

If a transform exists for two dimensional systems, a single conversion function

$$F = u'/u = m/m' \tag{14}$$

can be given for all units of the corresponding self-consistent unit systems. This function F may be written conveniently in the following form:

$$F = 10^{\sum_{i=1}^{n} \alpha_i \gamma_i} , \tag{15}$$

where the α_j's are defined by

$$\alpha_j = \log_{10} [u'(b_j)/u(b_j)] . \tag{16}$$

The basic quantities b_j are those of the umprimed system. One must always choose $n \geqq n'$. The equation $F^{-1} = u/u' = m'/m$ is valid for all units and measures even in those cases where no inverse dimensional transform is defined.

The conversion function from the *MKS*-system with the electrical units ampere, volt, coulomb, etc. and the mechanical units kilogram, meter, second, which can be regarded as based on *QMLT*, to the electromagnetic *cgs*-system *emu* based on *MLT* is given by

$$\log F = \gamma_Q - 3\gamma_M - 2\gamma_L \ ; \quad Fu(MKS) = u(cgs_{emu}) \ldots \tag{17}$$

where the γ's are those of the dimensional system *QMLT*. The *emu*-units for the magnetic field and the magnetomotive force must be taken as abamp/cm and abampere, respectively. No unique transformation can be given for the original Gaussian system since 1 oersted $= (1/4\pi)$ abamp/cm, and 1 gilbert $= (1/4\pi)$ abampere.

The *MKS*-units are transformed into the electrostatic *esu*-units of the *cgs*-system (statvolt, statcoulomb, etc.) by

$$\begin{aligned} \log F &= -(\log \tfrac{c}{10})\gamma_Q - 3\gamma_M - 2\gamma_L \\ &\approx -(9 + \log 3)\gamma_Q - 3\gamma_M - 2\gamma_L \end{aligned} \tag{18}$$

$$Fu(MKS) = u(cgs_{esu})$$

where the γ's are again those of the dimensional systems *QMLT*. $c = 2.997930 \times 10^{10}$ is the velocity of light in cm/sec.

The derivation of a complete set of dimensionless products

Let the solution of a physical problem depend on a set $q_1, q_2 \ldots q_s$ of quantities contained in a minimal physical subspace of n dimensions $(n < s)$. n of these quantities q can then be chosen as a new basis of DS_n. These n new basic quantities shall be denoted as b_j' $(j = 1, 2, \ldots, n)$. The only restriction on their choice is given by the condition that they must span DS_n. The remaining (*s-n*) quantities can now be written as linear combinations of these n new basic elements:

$$q_i(\gamma_{ij}') = \gamma_{i1} b_1' + \gamma_{i2} b_2' + \ldots + \gamma_{in} b_n' \ ; \ i = 1, 2, \ldots, (s\text{-}n). \tag{19}$$

Consequently:

$$\{\gamma_{i1} b_1' + \gamma_{i2} b_2' + \ldots + \gamma_{in} b_n'\} - q_i(\gamma_{ij}') = 0 . \tag{20}$$

Equation (20) defines all the vector sums of the quantities q_i $(i = 1, 2, \ldots, s)$ equal to zero which are possible in b_j' if trivial identities are not considered. The corresponding dimensionless products

$$\Pi_i = q_i^{-1} \prod_{j=1}^{n} b_j'^{\gamma_{ij}'} \ ; \quad i = 1, 2, \ldots, (s\text{-}n) \tag{21}$$

form, therefore, a complete set of dimensionless products of the quantities $q_1, q_2, \ldots, q_s$. Any physically homogeneous solution of the problem considered can thus be expressed as a function of these dimensionless products:

$$c = f(\Pi_1, \Pi_2, \ldots, \Pi_{s-n}) , \tag{22}$$

where c is a numeric constant usually of unit magnitude. Equations (21) and (22) are equivalent to Buckingham's theorem.

With another choice of the new basic quantities b_i', other complete sets of dimensionless products can be formed. The transformation of such sets shall not be discussed here, however, since a number of simple rules can be given to show how the set of dimensionless products appropriate for a particular solution can be found:

(1) Equation (22) is often desired in the form:

$$q_i \propto \prod_{j=1}^{n} b_j'^{\gamma_{ij}'} \psi(\Pi_2, \Pi_3, \ldots, \Pi_{s-n}) . \tag{23}$$

In such a case, the inclusion of q_1 in the new basis b_j' has to be avoided.

(2) Frequently, the functional dependence of q_1 on certain other quantities q_l, q_m, etc. is explicitly known or can be determined by simple experiments. It is then advantageous to include these quantities $q_l, q_m, \ldots$ in the basis b_j' since this permits a determination of the function ψ in the most economical way. It is in this way often possible to derive a formula explicitly by dimensional reasoning, (the only remaining uncertainty being, of course, a numeric factor frequently of unit magnitude).

(3) The dependence of q_1 on a quantity q_r may only be small. If q_r does not belong to b_j', it may be easily neglected later on by dropping the single dimensionless product Π_r.

(4) There may be some doubt if a quantity q_s or a quantity q_t contributes to a change of q_1. If these quantities q_s and q_t are not included in the basis b_j', their dimensionless products Π_s and Π_t can be easily exchanged in the complete formula. Such exchanges permit a certain experimenting in dimensional analysis.

Examples

1. The mass flow of a gas in a long tube of circular and uniform cross section may be considered in the molecular flow region where the inter-collisions of the gas molecules can be neglected against the collisions on the walls because of the long mean free path. The mass flow dG/dt depends evidently on the diameter d of the tube and its length l. The driving force can properly be expressed by the pressure difference Δp between both ends of the tube. Any differ-

ence in the behavior of various gases may be characterized by the molecular weight m of the gas under consideration. It is furthermore reasonable to assume that the flow depends on the density of the gas and is thus a function of the temperature. A convenient quantity to express this dependence is the product of gas constant and temperature $R\theta$. A dependence of the mass flow on the pressure itself is not considered since the mean free path is assumed to be large in comparison with the tube dimensions.

The following quantities are chosen as the new basis of DS:

$$\Delta p = M - L - 2T;\ m = M;\ R\theta = M + 2L - 2T \ .$$

The symbols M, L, T express the original basic quantities mass, length, time. A rearrangement of these equations yields:

$$M = m;\ L = \tfrac{1}{3}R\theta - \tfrac{1}{3}\Delta p;\ T = \tfrac{1}{2}m - \tfrac{1}{6}R\theta - \tfrac{1}{3}\Delta p \ .$$

The quantities $dG/dt = M - T$, $d = L$, and $l = L$ are now transformed into:

$$dG/dt = \tfrac{1}{2}m + \tfrac{1}{6}R\theta + \tfrac{1}{3}\Delta p \ ;\ d = \tfrac{1}{3}R\theta - \tfrac{1}{3}\Delta p \ ;$$
$$l = \tfrac{1}{3}R\theta - \tfrac{1}{3}\Delta p \ .$$

These equations correspond to Eq. (19). They can be rearranged to the form of Eq. (23) simply by inspection:

$$dG/dt \propto m^{1/2}(R\theta)^{1/6}(\Delta p)^{1/3}\phi\left(\frac{d(\Delta p)^{1/3}}{(R\theta)^{1/3}}\ ;\ \frac{l(\Delta p)^{1/3}}{(R\theta)^{1/3}}\right) .$$

This equation can be simplified by drawing the fairly obvious conclusion that the conductance of the tube, and therefore the mass flow, must be proportional to the inverse of the tube length. One obtains then:

$$dG/dt \propto m^{1/2}l^{-1}(R\theta)^{1/2}\psi\left(\frac{d(\Delta p)^{1/3}}{(R\theta)^{1/3}}\right) .$$

Not much more ambiguous, at least for the limited pressure range considered here, is the assumption that the mass flow is proportional to the pressure difference Δp. In case of any doubt, this could be checked, of course, by experiment. This assumption yields:

$$dG/dt = c\cdot\Delta p\sqrt{\frac{m}{R\theta}}\,\frac{d^3}{l}\ \ [\text{kg sec}^{-1}] \ ,$$

with a dimensionless numeric constant c of unit magnitude. (A full analytic treatment of this problem yields $c = \sqrt{\pi/18} \approx 0.42$.)

The remarkable result of this dimensional analysis is the proportionality of the mass flow to the third power of the tube diameter, a result which has been obtained without any consideration of the actual collision mechanism.

2. What is the space-charge-limited current of a magnetically focussed beam of charged particles in an accelerating electrical field?[8] For brevity, only the dimensional argument will be presented here. The following quantities seem to enter into the problem: the current I, the beam radius a, the magnetic field component in the direction of the beam B, the dielectric constant ϵ, the charge-to-mass ratio of the particles e/m, and the voltage V. In $QMLT$, these quantities are represented by the following vectors:

$$I = Q - T;\quad a = L;\quad B = -Q + M - T;$$
$$\epsilon = 2Q - M - 3L + 2T;\ e/m = Q - M;\ V = -Q + M + 2L - 2T.$$

B, ϵ, e/m, and V may be chosen as the new basic quantities. These vectors yield the following equations of transformation:

$$\left.\begin{aligned} B &= -Q + M - T \\ \epsilon &= 2Q - M - 3L + 2T \\ e/m &= Q - M \\ V &= -Q + M + 2L - 2T \end{aligned}\right\} \quad \begin{aligned} Q &= -B + \epsilon - \frac{1}{2}\frac{e}{m} + \frac{3}{2}V \\ M &= -B + \epsilon - \frac{3}{2}\frac{e}{m} + \frac{3}{2}V \\ L &= -B \quad - \frac{1}{2}\frac{e}{m} + \frac{1}{2}V \\ T &= -B \quad - \frac{e}{m} \end{aligned}$$

I and a are transformed by these equations to:

$$I = \epsilon + \frac{1}{2}\frac{e}{m} + \frac{3}{2}V \ ;\quad a = -B - \frac{1}{2}\frac{e}{m} + \frac{1}{2}V \ .$$

The saturation current is then given by:

$$I \propto \epsilon\sqrt{\frac{e}{m}}\ \sqrt{V^3}\phi\left(aB\sqrt{\frac{e}{m}}\Big/\sqrt{V}\right) .$$

The assumption that the current I is proportional to the cross section of the beam ($I \propto a^2$) leads to the final result:

$$I \propto \epsilon\sqrt{\left(\frac{e}{m}\right)^3}\ \sqrt{V}\ a^2B^2$$

Sequences of dimensional vectors

Table 1 shows clearly that the physical quantities can be arranged in a unique order according to the digital values of their dimensional vectors. This order depends, of course, on the particular dimensional system chosen. The same order principle can be extended to physical equations provided that in each equation the dimensional vectors are already arranged in an appropriate sequence. The following example gives the (decreasing) sequential order of four equations selected at random, representing

a) the potential of a dipole,
b) Maxwell's current equation,
c) a charged particle in a magnetic field, and
d) the Clausius-Clapeyron equation:

a) $V = Ql\cos\delta/4\pi\epsilon r^2$	$2\underline{13}2$	$\underline{1}12\underline{2}$	1000	10	10	10	α
b) $V = \oint Edl = -\int(\partial B/\partial t)ds$	$\underline{1}12\underline{2}$	$\underline{1}11\underline{2}$	$\underline{1}10\underline{1}$	20	10	1	
c) $d(mv)/dt = (e/2\pi R)d\phi_m/dt$	$\underline{1}12\underline{1}$	1000	100	$1\underline{1}$	10	1	1
d) $\Delta Q = \Delta V\cdot T\cdot dp/dT$	$12\underline{2}$	$1\underline{12}$	30	0,1	0,1		

The quantities are expressed in $QMLT,\theta$.

If such a sequence of equations encompassing a sufficiently large field of physical experience is transferred to punch cards, a digital computer can sort out all equations containing a certain combination of physical quantities. By developing a dimensional analysis from the resulting subset of physical equations, the dimensional argument can be based more easily on past physical experience than is presently the case. The examination of such a subset of equations may often enable one to visualize links between different fields which are not yet fully exploited. This concept of analyzing groups of equations promises to be useful in analogy considerations, in the conception of new devices, and in the evaluation of device systems.

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Table 1a **Basic, dimensionless, kinematic, and some thermal quantities**

Symbol	*Exponents of* Q	M	L	$T,$	θ		*Unit rationalized mksa*	*Quantitiy*	*Remarks*
$\alpha, \beta, \gamma, \theta, \vartheta, \zeta$				0		α	rad (radian)	plane angle	1 rad corresponds to circular arc of length of circle radius
ω, Ω				0		ω	sr (steradian)	solid angle	1 *sr* corresponds to 1/4 of surface area of sphere of unit radius
T, θ				0,	1		°K (degree abs.)	temperature	
t				1			s (second)	time	
l, b, h, x, y, z			1	0			m (meter)	length, breadth, height	
m		1	0	0			kg (kilogram)	mass	
Q	1	0	0	0			C (coulomb)	charge (electric)	
							Dimensionless Constants and Quantities		
ϵ				0				strain	$\Delta l/l$, $\Delta v/v$, et cetera
σ				0				Poisson's ratio	$\dfrac{\text{lateral contraction strain}}{\text{linear strain of rod}}$
ϵ_r				0				relative permittivity	ϵ/ϵ_0
χ				0				electrical susceptibility	$\chi = P/\epsilon_0 E$
α				0		α	rad	plane angle	
ω				0		ω	sr	solid angle	
							Kinematic (and some Thermal) Quantities		
T, θ				0,	1		°K	temperature	
ν, f				$\underline{1}$			Hz (hertz)	frequency	cycles/second
k				$\underline{1}$			s^{-1}	specific reaction rate const.	$k=(1/t)\ln(a/a\text{-}x)$
ω				$\underline{1}$		α	s^{-1}	angular velocity	$d\zeta/dt$
t				1			s	time	
T				1			s	period	
α				$\underline{2}$		α	s^{-2}	angular acceleration	$d^2\zeta/dt^2$
$\sigma, \tilde{\nu}$			$\underline{1}$	0			m^{-1}	wave number	$1/\lambda$
ρ			$\underline{1}$	0		α	m^{-1}	curvature	1/radius
l			1	0			m	length	
λ			1	0			m	wave length	
grad θ			$\underline{1}$	0,	1		m^{-1} °K	temperature gradient	
v			1	$\underline{1}$			ms^{-1}	velocity, speed	ds/dt
c			1	$\underline{1}$			ms^{-1}	velocity of light	$2.998 \times 10^8\ ms^{-1}$ in vacuo
a			1	$\underline{2}$			ms^{-2}	acceleration	d^2s/dt^{-2}
g			1	$\underline{2}$			ms^{-2}	accleration of gravity	$g=9.80665 ms^{-2}$
A, s			2	0			m^2	area	
D			2	$\underline{1}$			m^2s^{-1}	diffusion coefficient	flux $=-D$ grad c, $\partial c/\partial t = D\partial^2 c/\partial x^2$
ν			2	$\underline{1}$			m^2s^{-1}	kinematic viscosity	viscosity/density
C_m			2	$\underline{2}$,	$\underline{1}$		$\mathcal{J}kg^{-1}$ °K^{-1}	heat capacity (mass)	
V, v			3	0			m^3	volume	
U			3	$\underline{1}$			m^3s^{-1}	volume velocity	

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Table 1b **Mechanical and some thermal quantities**

Symbol	*Q*	*M*	*L*	*T*,	θ		*Unit rationalized mksa*	*Quantity*	*Remarks*
m		1	0	0			kg	mass	
R_M		1	0	$\underline{1}$			$Nm^{-1}s$	mechanical rectilinear resis.	R_M = force/velocity
C_M		$\underline{1}$	0	2			$N^{-1}m$	rectilinear compliance	C_M = change of length/force
γ, σ		1	0	$\underline{2}$			$\mathcal{J}m^{-2}$	surface tension	energy/area
I		1	0	$\underline{3}$			Wm^{-2}	sound intensity	$I = p^2/\rho_0 c$
S		1	0	$\underline{3}$			Wm^{-2}	Poynting vector	power/area
η		1	$\underline{1}$	$\underline{1}$			$Nm^{-2}s$	viscosity	η = shearing stress/velocity grad. $\perp$ to flow
p		1	1	$\underline{1}$			mkg s^{-1}	momentum	$p = mv$
$\mathcal{J}$		1	1	$\underline{1}$			mkg s^{-1}	impulse	$\mathcal{J} = \int_t F dt = \Delta p$
ω_e		1	$\underline{1}$	$\underline{2}$			$\mathcal{J}m^{-3}$	energy density	energy/volume
p		1	$\underline{1}$	$\underline{2}$			Nm^{-2}	pressure	force/area
s, σ		1	$\underline{1}$	$\underline{2}$			Nm^{-2}	stress	force/area
E, Y		1	$\underline{1}$	$\underline{2}$			Nm^{-2}	modulus of elasticity (Young's modulus)	$\sigma = E\Delta l/l$
k		1	$\underline{1}$	$\underline{2}$			Nm^{-2}	bulk modulus	Compression stress/volume strain
$T.S.$		1	$\underline{1}$	$\underline{2}$			Nm^{-2}	tensile strength	P/A_0 (max. load/orig. cross section)
μ		1	$\underline{1}$	$\underline{2}$			Nm^{-2}	shear modulus	shearing stress/shear strain
$S.S.$		1	$\underline{1}$	$\underline{2}$			Nm^{-2}	shear strength	max. shear load/cross section
F		1	1	$\underline{2}$			N (Newton)	force	mass $\times$ acceleration
C_V		1	$\underline{1}$	$\underline{2}$,	$\underline{1}$		$\mathcal{J}m^{-3}\,{}^\circ K^{-1}$	heat capacity (volume)	
λ		1	1	$\underline{3}$,	$\underline{1}$		$\mathcal{J}m^{-1}s^{-1}\,{}^\circ K^{-1}$	thermal conductivity	$\lambda = \Delta Ql/\text{area}\ t\theta$
$I, \mathcal{J}$		1	2	0			m^2kg	moment of inertia	$\int r^2 dm$
A		1	2	$\underline{1}$			$\mathcal{J}s$	action	$A = \int t \sum p_i \dot{q}_i dt$
R_R		1	2	$\underline{1}$		α	Nms	mech. rotational resis.	R_R = torque/angular velocity
L		1	2	$\underline{1}$		α	m^2kgs^{-1}	angular momentum	$L = r \times (mv)$
L		1	2	$\underline{1}$		α	m^2kgs^{-1}	moment of momentum	$L = \sum m_i (r \times \dot{r}_i)$
C_R		$\underline{1}$	$\underline{2}$	2		α	$N^{-1}m^{-1}$	rotational compliance	$M = \Delta\varphi/C_R$
E, U		1	2	$\underline{2}$			$\mathcal{J}$ (joule)	energy, work	(volt $\times$ coulomb) Potential energy: V, E_p; kin. energy: T, E_k
Q, q		1	2	$\underline{2}$			$\mathcal{J}$	quantity of heat	
M		1	2	$\underline{2}$		α	Nm	moment, torque	$M = r \times F$
S		1	2	$\underline{2}$,	$\underline{1}$		$\mathcal{J}\,{}^\circ K^{-1}$	entropy	for infinitesimal, revers. process: $dS = dq_{rev}/T$
R		1	2	$\underline{2}$,	$\underline{1}$		$\mathcal{J}\,{}^\circ K^{-1}(\text{mole}^{-1})$	molar gas constant	$R = (M/g)(pV/T) = 8.31662\mathcal{J}\,{}^\circ K^{-1}$
P		1	2	$\underline{3}$			W (watt)	power	work/time
U		1	2	$\underline{3}$		ω	Wsr^{-1}	radiation intensity	power/solid angle
r		$\underline{1}$	3	0			m^3kg^{-1}	specific refraction	$r = (1/\text{density})\ ((n^2-1)/(n^2+2))$ mol. refr. = sp. refr. $\times$ mol. weight
d		1	$\underline{3}$	0			$m^{-3}kg$	density	
M		1	$\underline{4}$	0			$m^{-4}kg$	inertance	$\rho = MdU/dt$, U = vol. velocity
Z_A		1	$\underline{4}$	$\underline{1}$			$Nm^{-5}s$ (mks acoustic ohm)	acoustic impedance	$Z_A = p/U$ (pressure/vol. velocity)
R_A		1	$\underline{4}$	$\underline{1}$			mks ac. ohm	acoustic resistance	real component of impedance
X_A		1	$\underline{4}$	$\underline{1}$			mks ac. ohm	acoustic reactance	imaginary component of impedance
C_A		$\underline{1}$	4	2			$N^{-1}m^5$	acoustic capacitance	$p = \Delta V/C_A$, V = volume

Table 1c **Electrical and magnetic quantities**

Symbol	Exponents of Q	M	L	T,	θ		Unit rationalized mksa	Quantity	Remarks
Q	1	0	0	0			C (coulomb)	charge (electric)	$1C$ = quan. of elec. transferred thru cross section in $1s$ by $1A$
ψ	1	0	0	0			C	flux (electric)	$Q=\int\int D.ds$ (same dimension as charge)
$\mathcal{J}$, i	1	0	0	$\underline{1}$			A (ampere)	current	$1A$ produces betw. 2 conductors $1m$ apart a force of 2×10^{-7} N per meter of length
mmf, $\mathfrak{F}$	1	0	0	$\underline{1}$		α	ampere turn	magnetomotive force	$\int Hdl$
ρ_L	1	0	$\underline{1}$	0			Cm^{-1}	linear charge density	charge/length
p	1	0	1	0			Cm	dipole moment	$p=ql$
H	1	0	$\underline{1}$	$\underline{1}$			Am^{-1}	magnetic field intensity	$H=B/\mu$
M	1	0	$\underline{1}$	$\underline{1}$			Am^{-1}	magnetization	magn. moment/volume
K	1	0	$\underline{1}$	$\underline{1}$			Am^{-1}	sheet current density	current/length
Q_m, q_m	1	0	1	$\underline{1}$			Am	pole strength	$\int\int\int \rho_m dv$
D	1	0	$\underline{2}$	0			Cm^{-2}	electric displacement	charge/area (flux density)
P	1	0	$\underline{2}$	0			Cm^{-2}	polarization	dipole moment /volume
$\mathcal{J}$	1	0	$\underline{2}$	$\underline{1}$			Am^{-2}	current density	current/area
ρ_m	1	0	$\underline{2}$	$\underline{1}$			Am^{-2}	pole density	pole strength/volume
m	1	0	2	$\underline{1}$			Am^2	magnetic (dipole) moment	mechanical moment/magnetic flux density
ρ	1	0	$\underline{3}$	0			Cm^{-3}	charge density (volume)	charge/volume $=\nabla\cdot D$
B	$\underline{1}$	1	0	$\underline{1}$			Wbm^{-2}	magnetic induction, magn. flux density	$B=\mu H$
A	$\underline{1}$	1	1	$\underline{1}$			$Wbm^{-1}(NA^{-1})$	vector potential	current × permability
E	$\underline{1}$	1	1	$\underline{2}$			Vm^{-1}	elect. field intensity	potential/length
Φ	$\underline{1}$	1	2	$\underline{1}$			Wb (weber)	magnetic flux	$\int\int Bds$
m	$\underline{1}$	1	2	$\underline{1}$			Wb	quantity of magnetism	
Λ	$\underline{1}$	1	2	$\underline{1}$		α	weber turns	flux linkage	flux × turns
V, ϕ	$\underline{1}$	1	2	$\underline{2}$			V (volt)	potential (electric)	$1V$ = potential betw. 2 points of a conductor carrying $1A$, when the power dissipation between the points is $1W$, there being no source of *emf* between the points
V, *emf*	$\underline{1}$	1	2	$\underline{2}$			V	electromotive force	$\int Edl$
M	$\underline{1}$	1	3	$\underline{1}$			Wbm	magnetic moment	
μ	$\underline{2}$	1	1	0			$\Omega m^{-1}s$	permeability	$c=1/\sqrt{\epsilon_0\mu_0}$, $\mu_0=4\pi\times10^{-7}$ henry/meter
L, M	$\underline{2}$	1	2	0			H (henry)	coefficient of inductance	
$\mathcal{P}$	$\underline{2}$	1	2	0			$A^{-1}Wb$	permeance	magnetic flux/*mmf*
$\mathcal{R}$	2	$\underline{1}$	$\underline{2}$	0			A Wb^{-1}	reluctance	*mmf*/magnetic flux
Z	$\underline{2}$	1	2	$\underline{1}$			Ω (ohm)	impedance (electric)	potential/current (V/A)
R	$\underline{2}$	1	2	$\underline{1}$			Ω	resistance (electric)	real component of impedance
X	$\underline{2}$	1	2	$\underline{1}$			Ω	reactance (electric)	imag. component of impedance
Y	2	$\underline{1}$	$\underline{2}$	1			mho (Siemens, Ω^{-1})	admittance (electric)	$Y=1/Z=1/(R+iX)=G-iB$
G	2	$\underline{1}$	$\underline{2}$	1			mho	conductance	$G=R/(R^2+X^2)$
B	2	$\underline{1}$	$\underline{2}$	1			mho	susceptance	$B=X/(R^2+X^2)$
C	2	$\underline{1}$	$\underline{2}$	2			F (farad)	capacitance	charge/potential
ρ	$\underline{2}$	1	3	$\underline{1}$			Ωm	resistivity	resistance × cross section/length
γ, σ	2	$\underline{1}$	$\underline{3}$	1			$\Omega^{-1}m^{-1}$	conductivity	$1/\rho$
ϵ	2	$\underline{1}$	$\underline{3}$	2			$CV^{-1}m^{-1}$ (farad/meter)	permittivity, dielectric constant	$\epsilon=D/E$ $\epsilon_0=8.854\times10^{-12}$ (farad/meter)

Table 2 **Alphabetic index of the physical quantities**

Quantity	*Symbol*	*Dim. No.*
acceleration	a	$1\underline{2}$
action	A	$12\underline{1}$
acoustic capacitance	C_A	$\underline{1}42$
acoustic impedance	Z_A	$1\underline{41}$
acoustic reactance	X_A	$1\underline{41}$
acoustic resistance	R_A	$1\underline{41}$
admittance (electric)	Y	$2\underline{12}1$
angle (plane)	α	0α
angle (solid)	ω, Ω	0ω
angular acceleration	α	$\underline{2}\alpha$
angular momentum	L	$12\underline{1}\alpha$
angular velocity	ω	$\underline{1}\alpha$
area	A, S	20
bulk modulus	k	$1\underline{12}$
capacitance	C	$2\underline{12}2$
charge (electric)	Q	1000
charge density (volume)	ρ	$10\underline{3}0$
coefficient of inductance	L, M	$\underline{2}120$
conductance	G	$2\underline{12}1$
conductivity	γ, σ	$2\underline{13}1$
current	$\mathfrak{I}$, i	$100\underline{1}$
current density	$\mathcal{J}$	$10\underline{21}$
curvature	ρ	$\underline{1}0\alpha$
density	d	$1\underline{3}0$
diffusion coefficient	D	$2\underline{1}$
dipole moment (electric)	P	1010
elasticity, modulus of	E, Y	$1\underline{12}$
electrical displacement (flux density)	D	$10\underline{2}0$
electric field intensity	E	$\underline{1}11\underline{2}$
electrical susceptibility	χ	0
electromotive force	V, emf	$\underline{1}12\underline{2}$
energy, work	E, U	$12\underline{2}$
energy density	ω_e	$1\underline{12}$
entropy	S	$12\underline{2},\underline{1}$
flux (electric)	ψ	1000
flux linkage	Λ	$\underline{1}12\underline{1}\alpha$
force	F	$11\underline{2}$
frequency	ν, f	$\underline{1}$
gas constant (molar)	R	$12\underline{2},\underline{1}$
gravity, acceleration of	g	$1\underline{2}$
heat, quantity of	Q, q	$12\underline{2}$
heat, capacity (mass)	C_m	$2\underline{2},\underline{1}$
heat, capacity (volume)	C_v	$1\underline{12},\underline{1}$
impedance	Z	$\underline{2}12\underline{1}$
inertance	M	$1\underline{4}0$
kinematic viscosity	ν	$2\underline{1}$
length	l	10
linear charge density	ρ_L	$10\underline{1}0$
magnetic dipole moment	m	$102\underline{1}$
magnetic field intensity	H	$10\underline{11}$
magnetic flux	Φ	$\underline{1}12\underline{1}$
magnetic induction (flux density)	B	$\underline{1}10\underline{1}$
magnetic moment	M	$\underline{1}13\underline{1}$
magnetism, quantity of	m	$\underline{1}12\underline{1}$
magnetization	M	$10\underline{11}$
magnetomotive force	mmf, $\mathfrak{F}$	$100\underline{1}\alpha$
mass	m	100
mechanical rectilinear resis.	R_M	$10\underline{1}$
mechanical rotational resis.	R_R	$12\underline{1}\alpha$
moment (torque)	M	$12\underline{2}\alpha$
moment of inertia	I, $\mathfrak{I}$	120
moment of momentum	L	$12\underline{1}\alpha$
momentum	p	$11\underline{1}$
period	T	1
permeability	μ	$\underline{2}110$
permeance	$\mathcal{P}$	$\underline{2}120$
permittivity (dielectric constant)	ϵ	$2\underline{13}2$
Poisson's ratio	σ	0
polarization	P	$10\underline{2}0$
pole density	ρm	$10\underline{21}$
Pole strength	Q_m, q_m	$101\underline{1}$
potential (electric)	V, ϕ	$\underline{1}12\underline{2}$
power	P	$12\underline{3}$
Poynting vector	S	$10\underline{3}$
pressure	p	$1\underline{12}$
radiation intensity	U	$12\underline{3}\omega$
reactance	X	$\underline{2}12\underline{1}$
rectilinear compliance	C_M	$\underline{1}02$
relative permittivity	ϵ_r	0
reluctance	$\mathcal{R}$	$2\underline{12}0$
resistance (electric)	R	$\underline{2}12\underline{1}$
resistivity (volume)	ρ	$\underline{2}13\underline{1}$
rotational compliance	C_R	$\underline{12}2\alpha$
shear modulus (rigidity)	μ	$1\underline{12}$
shear strength	$S.S.$	$1\underline{12}$
sheet current density	k	$10\underline{11}$
sound intensity	I	$10\underline{3}$
specific reaction rate const.	κ	$\underline{1}$
specific refraction	r	$\underline{1}30$
strain	ϵ	0
stress	s, σ	$1\underline{12}$
surface tension	γ, σ	$10\underline{2}$
susceptance	B	$2\underline{12}1$
temperature	T, θ	0,1
temperature gradient	grad θ	$\underline{1}0,1$
tensile strength	$T.S.$	$1\underline{12}$
thermal conductivity	λ	$11\underline{3},\underline{1}$
time	t	1
vector potential	A	$\underline{1}11\underline{1}$
velocity, speed	v	$1\underline{1}$
velocity of light	c	$1\underline{1}$
viscosity	η	$1\underline{11}$
volume	v, V	30
volume velocity	U	$3\underline{1}$
wavelength	λ	10
wave number	s, $\tilde{\nu}$	$\underline{1}0$