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Synthesis of Switching Functions by Linear Graph Theory

Abstract: Techniques of linear graph theory are applied to the study of switching networks. The first part treats the relationships among paths and circuits in a graph which will give a simple method of analyzing switching networks. The necessary conditions are given for the realizability of switching networks consisting of the specified elements. The second part is the synthesis which is accomplished by the use of the decomposition of cut-set matrices.

Symbols and definitions

C — Collection of all possible circuit sets of a graph including the empty set.

Circuit matrix — Matrix representing a collection of independent circuits in a graph where the entry at i, j position is 1 if element j is in circuit i in the collection and is 0 otherwise.

Circuit set — Collection of all the elements in a circuit or in an element disjoint union of circuits.

CP_{ij} — Collection of all possible sets each of which can be expressed as $p_{ij} \oplus c_k$ where set p_{ij} is in P_{ij} and set c_k is in C. That is, a set obtained by the ring sum of any set in P_{ij} and any set in C is in CP_{ij}, and the set which can not be obtained by the ring sum of a set in P_{ij} and a set in C is not in CP_{ij}.

$C \cup CP_{ij}$ — Collection of all sets in CP_{ij} and in C.

Cut-set — Collection of elements in a connected graph having the properties that (1) by the deletion of all elements in the set, the graph will be divided into two connected subgraphs (including the case that a subgraph may consist of one isolated vertex) and (2) any proper subset of the set does not have Property (1).

Cut-set matrix — Matrix of order $v-1$ by e of rank $v-1$ representing independent cut-sets in a graph of v vertices and e elements where the entry at i, j position is 1 if element j is in cut-set i, and 0 otherwise.

Element — Line segment, together with its end-points.

Element disjoint union of circuits — Union of circuits which have no elements in common. (See Ref. 9.)

Incidence matrix (of a connected graph) — Matrix of order $v-1$ by e and of rank $v-1$ representing a connected graph of v vertices and e elements. The entry at i, j position is defined to be 1 if element j is incident at vertex i, and 0 otherwise.

Nonseparable graph — Graph which is not a separable graph.

Path between vertices i and j — A subgraph which has the following properties:
(1) The subgraph consists of at least one element.
(2) The subgraph is connected and contains the vertices i and j.
(3) When any element in the subgraph is removed, the remaining subgraph does not satisfy both (1) and (2).

Path set — Collection of all the elements in a path.

ϕ — Empty set.

P_{ij} — Collection of the path sets representing all possible paths between the fixed vertices i and j. Any set in P_{ij} thus represents a path between the vertices i and j in a graph, and there exists no path between the vertices i and j whose path set is not in P_{ij}.

Ring sum — An operation on sets u_1 and u_2, being the set consisting of elements in either u_1 or u_2 but not in both u_1 and u_2. The symbol $\oplus$ indicates the ring sum.

Separable graph — Connected graph which can be divided into two subgraphs which are connected by only one vertex. (See Ref. 9.)

Vertex — End-point of an element.

Introduction

Several papers[1-3] have shown the possibility of applying linear graph theory to the synthesis of switching functions. Löfgren[4] and Gould[5] give sufficient conditions for the existence of a switching network consisting of the specified elements without any restriction on the network structure.

The method of synthesizing a binary switching function shown in this paper consists of two parts. The first part is the necessary condition and the second part is the sufficient condition for the existence of a switching network consisting of the specified elements which satisfies a given switching function. The procedure used in the first part is similar to, but simpler than, those used by Löfgren and Gould. The second part is accomplished by the use of the necessary and sufficient conditions for realizability of cut-set matrices,[6] whereas Löfgren gives a part-by-part method of constructing a switching network and Gould shows a part-by-part method of forming a tree of a switching network which consists of the specified elements.

Switching function

A switching function can be considered as a representation of a series of conditions under which information can be transferred between two specified vertices in a binary switching network. This is easily seen by expressing a switching function F as a sum of terms f_i, each of which is a product form of switching variables as

$$F=\sum_{(i)} f_i\,. \tag{1}$$

Since F is 1 when f_i is 1 for any i, f_i represents a condition for the transmission of information.

The structure of a switching network can be represented by giving switching variables as the weights of elements in the graph. Then the weights of the elements in a path betwen vertices i and j of a graph give the conditions to transfer information between vertices i and j. Thus the product form of the weights of all elements in a path between vertices i and j of a graph can be a term f_i in Eq. (1).

Path sets

In a study of the properties of paths and circuits, it is convenient to use sets which are collections of elements, rather than to use subgraphs. A *path set* is the collection of all elements in a path and a *circuit set* is the collection of all elements in either a circuit or an element disjoint union of circuits.

The symbol C is the collection of all possible circuit sets in a connected graph. The empty set ϕ is also included in C. For example C, of the graph in Fig. 1 consists of the following sets:

ϕ, $\{ab\}$, $\{acd\}$, $\{ce\}$, $\{bcd\}$, $\{abce\}$, $\{ade\}$, and $\{bde\}$.

(Notice that set $\{abce\}$ represents an element disjoint union of circuits.)

The symbol P_{ij} represents the collection of all possible path sets, each of which represents a path between the fixed vertices i and j in a connected graph. For example, P_{ij} of the graph in Fig. 1 consists of the following sets:

$\{a\}$, $\{b\}$, $\{cd\}$, and $\{de\}$.

The symbol CP_{ij} represents the collection of sets which has the property that the ring sum of any set in P_{ij} and any set in C is in CP_{ij}, and there exist no sets in CP_{ij} which can not be expressed as a ring sum of a set in P_{ij} and a set in C. For example, CP_{ij} of the graph in Fig. 1 consists of the following sets:

$\{a\}$, $\{b\}$, $\{cd\}$, $\{de\}$, $\{ace\}$, $\{abcd\}$, $\{bce\}$, and $\{abde\}$.

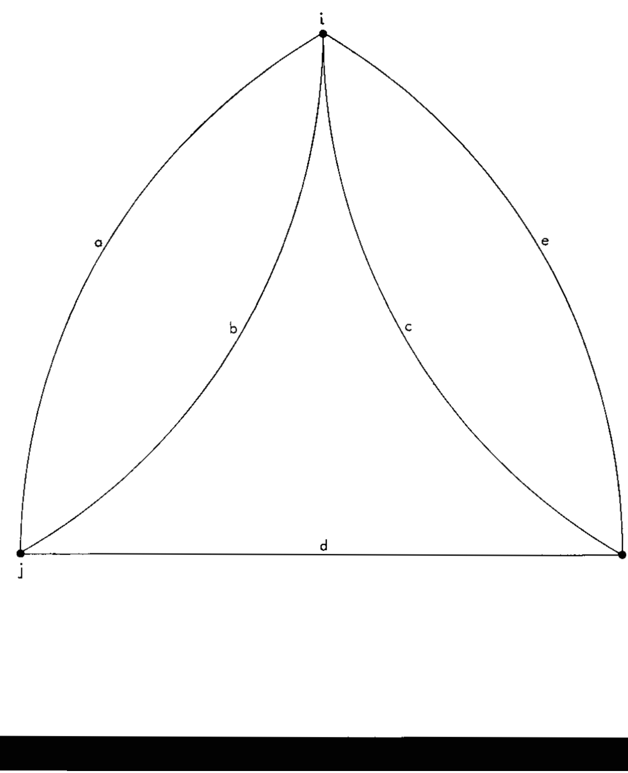
The subgraphs representing these sets are shown in Fig. 2. Notice that each subgraph is either a path between i and j or a path between i and j and a circuit which have no elements in common.

Whitney[6] shows that if sets s_1 and s_2 are circuit sets, the sets $s_1 \oplus s_2$ is also a circuit set. This result can be extended by restricting sets s_1 and s_2 as follows: Suppose s_1 and s_2 have at least one element in common. Let this element be h and the two vertices between which h is connected be i and j. By removing h from s_1 and s_2, two sets u_1 and u_2 can be obtained, that is, $u_1=s_1\oplus\{h\}$ and $u_2=s_2\oplus\{h\}$. By definition, u_1 and u_2 are in CP_{ij}. Notice that if s_1 and s_2 represent circuits but not element disjoint unions of circuits, u_1 and u_2 represent paths between vertices i and j. Since $s_1\oplus s_2=u_1\oplus u_2$, the following theorem can be obtained:

Theorem 1: If u_1 and u_2 are in CP_{ij} of a graph, then the set $c=u_1\oplus u_2$ is in the collection C of circuit sets.[8] Likewise,

Corollary 1: If u_1 is in CP_{ij} and c is in C, the set $u_1 \oplus c$ is in CP_{ij}. This is true because $u_1\oplus u_2\oplus u_1=u_1\oplus u_1\oplus u_2=u_2$.

Figure 1 **A connected graph.**



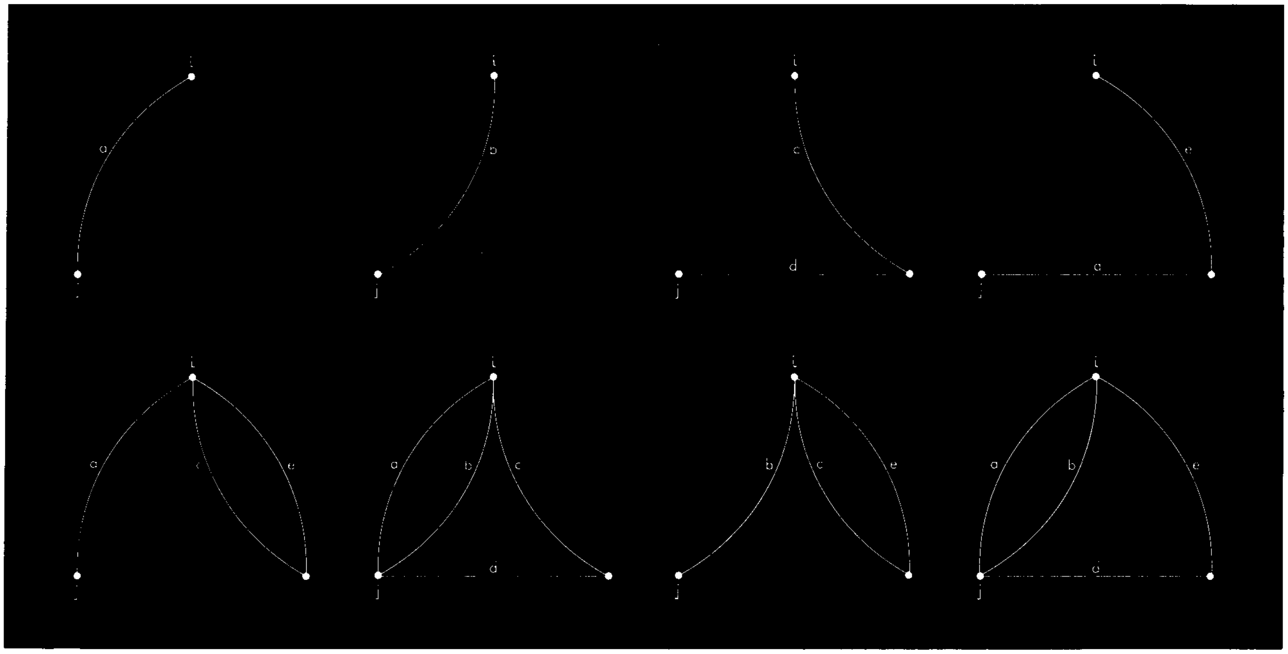


Figure 2 **All possible subgraphs corresponding to the sets in CP^{ij} of the graph in Fig. 1.**

It can be seen that the collection C of circuit sets in a graph is an Abelian group under the ring sum. Also by defining product of a set c in C and β, where β is in binary field B as

$$\beta c = c\beta = \begin{cases} c & \text{if } \beta = 1 \\ \phi & \text{if } \beta = 0 \end{cases} \tag{2}$$

C with B form a vector space. Thus there exist independent sets in C. Since a set in C represents a circuit, there exist $e-v+1$ independent sets in C of a connected graph consisting of e elements and v vertices.

The collection $C \cup CP_{ij}$ of sets, each of which is in either C or CP_{ij}, is also an Abelian group under the ring sum because of Theorem 1 and Corollary 1. Hence with a binary field B, $C \cup CP_{ij}$ form a vector space and there exist $e-v+2$ independent sets[8] in $C \cup CP_{ij}$. The reason is as follows: By inserting an element h between vertices i and j of a connected graph G, a new graph G' is obtained. It is clear that the sets in the collection C' of circuit sets in G' which do not contain element h are all the sets in the collection C of circuit sets in the given graph G. On the other hand, the sets in C' which contain element h are not the circuit sets in C of G: but without the element h, these sets are all sets in CP_{ij} of G. This means that by removing h from every set in C' which contains h, the new collection of sets is $C \cup CP_{ij}$. There exist $e'-v+1$ independent sets in C', where $e'(=e+1)$ is the number of elements in G' and e is the number of elements in G. Hence, there exist $e-v+2$ independent sets in $C \cup CP_{ij}$.

Since there exist only $e-v+1$ independent sets in C, at least one set in a collection of $e-v+2$ independent sets in $C \cup CP_{ij}$ belongs to CP_{ij}. Furthermore, all sets in a collection of $e-v+2$ independent sets in $C \cup CP_{ij}$ can be in CP_{ij}, because the ring sum of any set in C and any set in P_{ij} is in CP_{ij} and $P_{ij} \subset CP_{ij}$.

Representation of elements in switching networks

In the previous section, the properties of sets in P_{ij}, C, and CP_{ij} are discussed. It must be noticed that these properties hold only when different symbols are used to represent different elements in a graph. On the other hand, more than one element in a binary switching network may represent the same switching variable. Also, it is convenient to represent each element in a switching network by its weight. Hence, hereafter the elements in a graph are represented by their weights, which are the switching variables, and the symbols used to represent switching variables are as follows:

Type 1: To distinguish the different elements controlled under the same conditions, subscripts are used, as y_0, y_1, y_2, et cetera. Subscript 0 of y_0 can be omitted.

Type 2: To indicate two elements which are *complementary* to each other, a bar is used, as x_i and $\bar{x}_i$; that is, whenever x_i is 1, $\bar{x}_i$ is 0 and whenever $\bar{x}_i$ is 1, x_i is 0.

As an example of the elements in Type 1, a relay consisting of multiple contacts is shown in Fig. 3a and its graphical representation in Fig. 3b. Notice that whenever one of $y_1, y_2, \ldots,$ and y_k is 1, all of them are 1, and whenever one of $y_1, y_2, \ldots,$ and y_k is 0, all of them are 0.

For convenience, the *set-product* of a set is defined as the product of all elements in the set. It has been shown that the elements (represented by their weights) in a path between the two specified vertices in a graph represent a

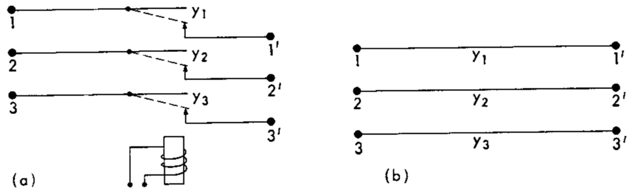


Figure 3 **A relay consisting of multiple contacts (a), and the graphical representation of the relay (b).**

condition under which information will be transmitted between these vertices.

If a path contains two elements which are complementary to each other, it is called an *open-path* and a signal will not be transferred by the path under any condition. A path set representing an open-path is called an *open-path set* and P_{ij} of a graph may contain open-path sets. It can be seen that the set-product of an open-path set is zero. Hence, the set-product of a path set can be considered as a term f of a switching function F in Eq. (1) and summation of the set-product of all the path sets in P_{ij} of a graph G will represent a switching function F of G between the vertices i and j, i.e.,

$$F=\sum \text{set-product of all the sets in } P_{ij}\,. \tag{3}$$

In addition to the identities of Boolean algebra, the following identity holds because of the representation of elements:

$$x_q=x_r \text{ for any } q \text{ and } r\,. \tag{4}$$

Example: To obtain the switching function from the graph in Fig. 4 between vertices i and j, the following procedure can be used.

Since it is only necessary to know $e-v+2$ independent sets in $C\cup CP_{ij}$ of the graph to find all sets in $C\cup CP_{ij}$, we form $e-v+2=4$ independent sets from the graph as $c_1=\{a_1e_1b\}$, $c_2=\{b\bar{b}d\}$, $c_3=\{de_2a_2\}$, and $p=\{a_1\bar{b}e_2\}$.

The designations c_1, c_2, and c_3 are circuit sets and p is a path set in the graph. (It is not necessary to use $e-v+1$ independent sets in C. However, these are usually very simple to obtain from a graph.[8,9]) By Theorem 1 and Corollary 1, the following sets are all sets in CP_{ij}:

$$\begin{array}{ll}
p=\{a_1\bar{b}e_2\} & c_1\oplus c_2\oplus p=\{de_1e_2\} \\
c_1\oplus p=\{b\bar{b}e_1e_2\} & c_1\oplus c_3\oplus p=\{a_2b\bar{b}de_1\} \\
c_2\oplus p=\{a_1bde_2\} & c_2\oplus c_3\oplus p=\{a_1a_2b\} \\
c_3\oplus p=\{a_1a_2\bar{b}d\} & c_1\oplus c_2\oplus c_3\oplus p=\{a_2e_1\}\,.
\end{array}$$

The switching function F can be written as the summation of the set-products of the above sets as

$$F=a_1\bar{b}e_2+b\bar{b}e_1e_2+a_1bde_2+a_1a_2\bar{b}d+de_1e_2+a_2b\bar{b}de_1+a_1a_2b+a_2e_1\,.$$

Then, by Eq. (4) and by the use of identities of Boolean algebra, F can be simplified as

$$F=ab+de+ae+ad\,.$$

Synthesis of switching functions by linear graph theory

The synthesis of a switching function by linear graph theory is the formation of a switching function of the form of Eq. (3) from a given switching function by the use of the identities of Boolean algebra and Eq. (4). To accomplish this procedure, a method is necessary to determine whether or not a function is of the form of Eq. (3).

For each term f_i in a switching function F in Eq. (1), we form a set consisting of all the elements in f_i. The collection of these sets is called "the collection of sets corresponding to F." The function F is equal to the set-product of all the sets in D if D is the collection of sets corresponding to F.

If F is of the form in Eq. (3), then the collection of sets corresponding to F should represent P_{ij} of a graph. By Theorem 1 and Corollary 1, the ring sums of an even number of any sets in P_{ij} of a graph is a circuit set which is the collection of elements in a circuit or an element disjoint union of circuits in the graph. However, the ring sum of an odd number of any sets in P_{ij} is a set in CP_{ij}.

In general, every term f_i in a switching function F in Eq. (1) is not always zero. Hence, when the collection of sets corresponding to F is formed from F, there exists no set containing two elements which are complementary to each other. However, there is a possibility of having open-paths in a graph. This means that F may not be of the form in Eq. (3), but may be of the form

$$F=\sum \text{set-product of sets in } P'_{ij}\,, \tag{5}$$

where P'_{ij} is the collection of all the sets in P_{ij} except open-path sets.

If the set obtained by ring sum of an odd number of sets in P'_{ij} is not in P'_{ij}, it must be either an open-path set or a set which is in CP_{ij}. (Notice that $P_{ij}\subset CP_{ij}$.) Hence, if the collection D of sets corresponding to a switching function is P'_{ij}, D has the property that every set in the collection E, which is obtained by ring sum of all possible combinations of odd number of sets in D, is one of the following three:

(1) The set is in D.

(2) The set can be decomposed as $u_p\oplus u_c$, where u_p is in E, u_c contains at least two elements, and $u_p\cap u_c=\phi$.

(3) If the set does not satisfy either (1) or (2), the set must contain two elements which complement each other.

If D does not have the above property, it shows that switching function, by which D is formed, is not of the form in Eq. (5). Hence, one must change the switching function by using the identities of Boolean algebra and by using Eq. (2). Even if D has the above property, it is not sufficient to say that the switching function, by which D is obtained, is of the form in Eq. (5).

Suppose the collection D of sets corresponding to a switching function F has the above property. We then add the path set u_t, which consists of one element t to the collection D, and form the collection C_t of sets by ring sum of all possible combinations of even number of sets in D, including u_t. It is clear that the collection C_t has the property that a set obtained by ring sum of any sets in C_t is in C_t. Hence, C_t is an Abelian group under the ring sum and, with a binary field B, C_t forms a vector space. Thus there exist independent sets in C_t. Let I be a collection of independent sets in C_t. Then a matrix $R=[r_{pq}]$ can be formed whose rows represent the sets in I and columns represent elements in the sets, where $r_{pq}=1$ if element q is in set p which is in I, and $r_{pq}=0$ otherwise.

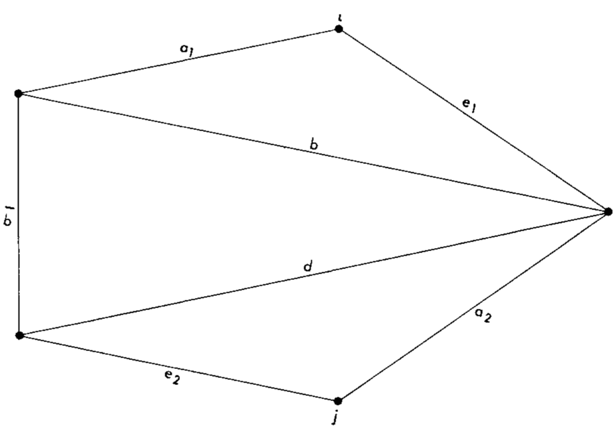
If R is a circuit matrix, we know that there exists a graph G which has the property that the collection of all the circuit sets of G is C_t. Since every set in C_t, which contains element t, is obtained by ring sum of $u_t=\{t\}$ and odd number of sets in P'_{ij}, one can obtain, by removing element t from G, the graph whose P'_{ij} is equal to the collection D of sets corresponding to the switching function F, where i and j are the vertices between which element t was connected. Hence, G without element t is a switching network which satisfies the given switching function.

At present, there is no simple method of testing whether or not a matrix is a circuit matrix. Although the following method for converting R_c to Q is not a completely definitive procedure, it is a very useful one since there exists an efficient method for determining whether Q is a fundamental cut-set matrix[7] (which is explained in the next section). It is known that by proper choice of independent sets in C_t, one can obtain a matrix R_c of the form

$$R_c=[UR_{c12}]\ . \tag{6}$$

Notice that choosing the different independent sets in C_t to form R_c is equivalent to premultiplying (modulo 2) R by a nonsingular matrix $T=[t_{nm}]\,(t_{nm}=1,0)$. From R_c in Eq. (6), one can obtain a matrix $Q=[R'_{c12}U]$ where R'_{c12} is the transpose of R_{c12}. It is known that if Q is a fundamental cut-set matrix, there exists a graph whose circuit matrix is R. Hence, one can find a network which satisfies the given switching function.

Figure 4 **A connected graph representing a switching network.**



Realizability conditions of cut-set matrices

An incidence matrix A_i of order $v-1$ by e and of rank $v-1$ indicates the structure of a graph G consisting of v vertices and e elements. The rows of A_i represent the vertices of G except one vertex called the reference vertex and the columns A_i represent the elements in G. The entry at i, j position of A_i is defined to be 1 if element j is connected to vertex i, and 0 otherwise. Hence, there exist at most two 1's in each column of A_i. A matrix of order $v-1$ by e and of rank $v-1$ can be an incidence matrix if every column has at most two 1's (all other entries in the column are 0).

A row of a cut-set matrix of order $v-1$ by e and of rank $v-1$ represents a cut-set (see the definition of cut-set) and a column of the cut-set matrix represents an element in a graph of v vertices and e elements. The entry at i, j position of a cut-set matrix is defined to be 1 if element j is in cut-set i and 0 otherwise. A fundamental cut-set matrix A_c is a cut-set matrix which has the form $A_c=[A_{c11}U]$, where U is a unit matrix.

Suppose a matrix $Q=[Q_{11}U]$, of order $v-1$ by e, is given. We form the matrix H from Q by deleting every column which has 1 at row i and by deleting row i. The rows and the columns of H are rearranged so that H can be partitioned, as

$$H=\left[\begin{array}{c|c} H_{11} & 0 \\ \hline 0 & H_{22} \end{array}\right]. \tag{7}$$

If it is impossible to partition H as in Eq. 7, we define that H_{11} of H consists of no rows and no columns and $H_{22}=H$.

Definition: When a matrix D is obtained from a matrix E by the deletion of rows and columns of E, the symbols to indicate the rows and the columns will follow. That is, row i (column j) of D is the row (the column) obtained from row i (column j) of E.

We form two matrices, $M_1(i)$ and $M_2(i)$, called a "pair of M-submatrices with respect to row i," by the following procedure:

$M_1(i)$ is obtained by deleting all rows and columns of H_{11} from the given matrix Q.

$M_2(i)$ is obtained by deleting all rows and columns of H_{22} from Q. The symbol i in the parenthesis of $M_1(i)$ and $M_2(i)$ indicates the row in the M-submatrices which is used to obtain H. (See the process of obtaining $M_1(4)$ and $M_2(4)$ from Q in the last section.)

The characteristics of a pair of M-submatrices $M_1(i)$ and $M_2(i)$ of Q with respect to row i are that

(1) every row except row i of Q will be in either $M_1(i)$ or $M_2(i)$ but not in both, and

(2) row i of Q will be in both $M_1(i)$ and $M_2(i)$.

Since an M-submatrix $M(i)$ has the form $M(i)=[M(i)_{11}U]$, one can obtain M-submatrices $M(ij)$ and $M(j)$ of $M(i)$ with respect to row $j\neq i$, by the same procedure as that used to obtain $M_1(i)$ and $M_2(i)$ from Q. Notice that the symbols indicating the rows in the parenthesis of an M-submatrix are the rows *in the submatrix* which are already used to obtain M-submatrices. By the characteristics of M-submatrices, only one of the M-submatrices of matrix $M(i)$ with respect to row $j \neq i$ contains row i, because row i is not the row used to obtain a pair of M-submatrices from $M(i)$. However, row j is in both of these M-submatrices. Hence, the inside of the parenthesis of one of these M-submatrices of $M(i)$ is i, j and that of the other is j. Thus the M-submatrices of $M(i)$ with respect to row j are $M(i, j)$ and $M(j)$.

When a row in an M-submatrix has been used (i.e., the symbol representing the row is already in the parenthesis of an M-submatrix), one can not use the row to obtain a pair of submatrices from the M-submatrix.

By continuing the above process until every row of every M-submatrix has been used, v M-submatrices of the form $M(i_1, i_2, \ldots, i_k)$ can be obtained where $M(i_1, i_2, \ldots, i_k)$, consists of rows $i_1, i_2, \ldots,$ and i_k. Such an M-submatrix is called a *minimum M-submatrix,* and the collection of these v minimum M-submatrices is called a *set of minimum M-submatrices.* (See the minimum M-submatrices in the last section.)

If one can partition H in the form of Eq. (7) more than one way, then there are more than one pair of M-submatrices with respect to a row. During the process of obtaining minimum M-submatrices, there may be more than one pair of M-submatrices of $M(i_1, i_2, \ldots, i_j)$ with respect to row i_n. If this is the case, for each pair of M-submatrices of $M(i_1, i_2, \ldots, i_j)$ with respect to row i_n, there exists a set of minimum M-submatrices.

Theorem 2: The necessary and sufficient condition that a matrix $Q = [Q_{11}U]$ be a fundamental cut-set matrix is that there exists a set of minimum M-submatrices of Q so that every minimum M-submatrix in the set is realizable as an incidence matrix (every column of every minimum M-submatrix in the set has at most two 1's).[7]

If a set of minimum M-submatrices satisfies Theorem 2, one can form a graph from these submatrices by the following procedure:

(1) We collect all minimum M-submatrices in the set, each of which has at least two rows.

(2) By considering each of these minimum M-submatrices as an incidence matrix, we can obtain the subgraph for each of these submatrices. The symbol to indicate a vertex in the subgraph corresponding to a minimum M-submatrix is the same symbol which indicates the row of the submatrix representing the vertex. The symbol for the reference vertex must be different for each subgraph. In these subgraphs, there are at most two vertices having the same symbol.

(3) We choose two subgraphs in which there are two vertices having the same symbol. Let these subgraphs be G_a and G_b. Also let K be the vertex in G_a and in G_b. For each element in G_a which is connected to vertex K in G_a there is an element which is connected to vertex K in G_b. Furthermore, the symbols for these two elements are the same. Let $e_1, e_2, \ldots, e_k$ be the elements connected to vertex K in G_a. Then there are the elements which are connected to vertex K in G_b and whose symbols are also $e_1, e_2, \ldots,$ and e_k. We remove all of these elements from G_a and G_b. Then we combine these two resultant subgraphs by connecting element e_p between two vertices, one of which is the vertex in G_a other than vertex K on which element e_p was connected. The other is the vertex in G_b other than vertex K, on which element e_p was connected for $p = 1, 2, \ldots, k$. By repeating this process, all subgraphs will be combined into one graph.

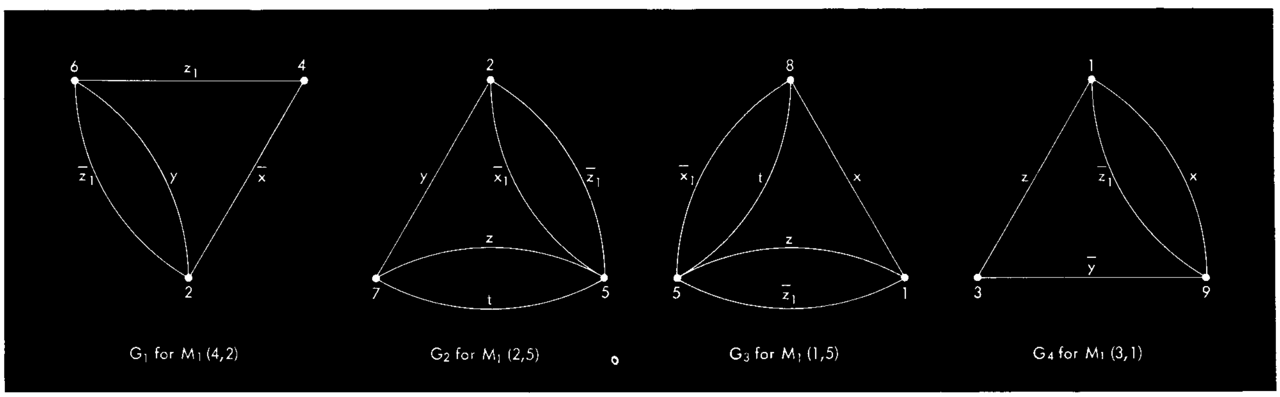
Example of synthesizing a switching function

From switching function $F = xy\bar{z} + x\bar{y}z + \bar{x}yz$, collection D of sets $u_1 = \{xy\bar{z}\}$, $u_2 = \{x\bar{y}z\}$, and $u_3 = \{\bar{x}yz\}$ can be obtained. Since $u_1 \oplus u_2 \oplus u_3 = \{\bar{x}\bar{y}\bar{z}\}$ is not in D, F is not of the form in Eq. (5). By changing F to $F' = xy\bar{z}_1 + x\bar{y}z + \bar{x}yz_1$, the collection D of sets consists of

$$u_1 = \{xy\bar{z}_1\},\ u_2 = \{x\bar{y}z\},\ \text{and}\ u_3 = \{\bar{x}yz_1\}\ .$$

Since $u_1 \oplus u_2 \oplus u_3$ contains z_1 and $\bar{z}_1$, D has the property of P'_{ij}. The collection C_t consists of sets $u_1 \oplus t$, $u_2 \oplus t$, $u_3 \oplus t$, $u_1 \oplus u_2$, $u_1 \oplus u_3$, $u_2 \oplus u_3$, and $u_1 \oplus u_2 \oplus u_3 \oplus t$, and by using the first three of these sets which are independent sets in C_t, matrix R can be formed as

Figure 5 **The graphs corresponding to the incidence matrices.**



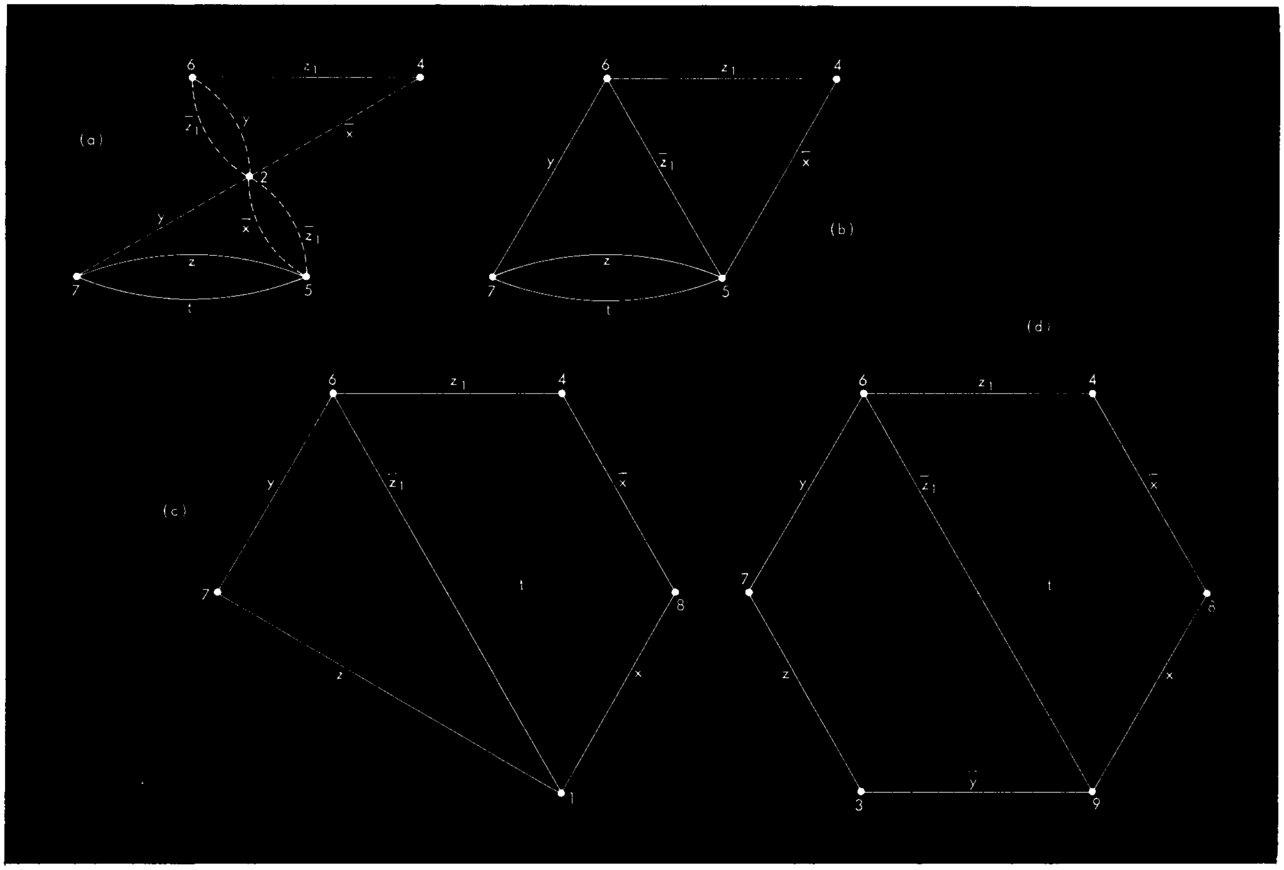


Figure 6 **(a) Graph obtained by removing elements $\bar{z}_1$, y, $\bar{x}$ from G_1 and G_2 in Fig. 5.**
(b) The graph obtained from G_1 and G_2 in Fig. 5.
(c) The graph obtained from the graph in (b) and G_3 in Fig. 5.
(d) The graph obtained from the graph in (c) and graph G_4 in Fig. 5.

$$R=\begin{array}{c} \begin{array}{cccccccc} \bar{z}_1 & z & \bar{x} & x & y & \bar{y} & z_1 & t \end{array} \\ \left[\begin{array}{cccccccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

Then, the matrix $Q=[R'_{12}U]$ is

$$Q=\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccccc} \bar{z}_1 & z & \bar{x} & x & y & \bar{y} & z_1 & t \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

The matrix H_a of Q with respect to Row 4 is

$$H_a=\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} \bar{z}_1 & z & x & y & \bar{y} & t \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array},$$

which can not be partitioned as in Eq. (7). Hence, H_{11} consists of no rows and no columns and $H_{22}=H$. The pair of M-submatrices are

$$M_1(4)=Q, \text{ and } M_2(4)=4\begin{array}{c} \begin{array}{cc} \bar{x} & z_1 \end{array} \\ [\begin{array}{cc} 1 & 1 \end{array}] \end{array}.$$

The matrix H of $M_1(4)$ with respect to Row 3 can not be partitioned as in Eq. (7). Hence, the M-submatrices of $M_1(4)$ with respect to Row 3 are

$$M_1(4,3)=Q, \text{ and } M_1(3)=3\begin{array}{c} \begin{array}{cc} z & \bar{y} \end{array} \\ [\begin{array}{cc} 1 & 1 \end{array}] \end{array}.$$

The matrix H_b of $M_1(4, 3)$ with respect to Row 2 can be partitioned as

$$H_b=\begin{array}{c} \\ 4 \\ 1 \\ 3 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} z_1 & z & x & \bar{y} & t \end{array} \\ \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

Hence, the M-submatrices of the matrix $M_1(4, 3)$ with respect to Row 2 are

$$M_1(3,2) = \begin{array}{rc} & \begin{array}{ccccccc} \bar{z}_1 & z & \bar{x} & x & y & \bar{y} & t \end{array} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array},$$

$$\text{and } M_2(4,2) = \begin{array}{rc} & \begin{array}{cccc} \bar{z}_1 & \bar{x} & y & z_1 \end{array} \\ \begin{array}{r} 2 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

By using Row 1, the pair of M-submatrices of matrix $M_1(2, 1)$ are

$$M_1(2,1) = \begin{array}{rc} & \begin{array}{cccccc} \bar{z}_1 & z & \bar{x} & x & y & t \end{array} \\ \begin{array}{r} 1 \\ 2 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array},$$

$$\text{and } M_1(3,1) = \begin{array}{rc} & \begin{array}{cccc} \bar{z}_1 & z & x & \bar{y} \end{array} \\ \begin{array}{r} 1 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Finally, by using Row 5, the pair of M-submatrices of matrix $M_1(2, 1)$ are

$$M_1(2,5) = \begin{array}{rc} & \begin{array}{ccccc} \bar{z}_1 & z & \bar{x} & y & t \end{array} \\ \begin{array}{r} 2 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array},$$

$$\text{and } M_1(1,5) = \begin{array}{rc} & \begin{array}{ccccc} \bar{z}_1 & z & \bar{x} & x & t \end{array} \\ \begin{array}{r} 1 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

The M-submatrices $M_2(4)$, $M_1(3)$, $M_1(4, 2)$, $M_1(3, 1)$, $M_1(2, 5)$, and $M_1(1, 5)$ are the minimum M-submatrices, and these matrices form the set of minimum M-submatrices of Q. Since all of these are realizable as incidence matrices, there exists a switching network consisting of elements $\bar{z}_1$, z, $\bar{x}$, x, y, $\bar{y}$, and z_1 which satisfies the given switching function.

To construct the switching network, we form the subgraphs corresponding to the matrices $M_1(4, 2)$, $M_1(3, 1)$, $M_1(2, 5)$, and $M_1(1, 5)$ as shown in Fig. 5.

To combine G_1 and G_2, we remove the elements $\bar{x}$, $\bar{z}_1$ and y from G_1 and G_2, as shown in Fig. 6a, because these elements are connected to vertex 2 which is the vertex in both G_1 and G_2. Then we connect $\bar{x}$ between vertices 4 and 5, $\bar{z}_1$ between vertices 6 and 5, and y between vertices 6 and 7 as shown in Fig. 6b. The resultant graph and G_3 can be combined by the elements $\bar{z}_1$, z, $\bar{x}$, and t as shown in Fig. 6c. The resultant graph (Fig. 6c) with G_4 gives the graph (Fig. 6d) from which the switching network can simply be obtained by removing the element t. The two specified terminals of the switching network are those on which t was connected.

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