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Synthesis of a Communication Net

Abstract: A systematic method is given for the realization of communication nets from their terminal capacity matrices. It is shown that this procedure results in a net whose total branch capacity is minimum for all nets satisfying the same terminal capacity matrix. It is also shown that when the terminal capacity matrix is indeterminate, then, for a given total branch capacity, the total terminal capacity is highest when all terminal capacities are made equal.

1. Introduction

Consider any conveying system with multiple terminals, such as a communication net, a power distribution system, a highway network, or any one of many other possibilities. Topologically, their structures are equivalent. Each of them can be represented by a graph, the nodes of which correspond to the terminals, while the connections between the terminals are represented by branches with weight. The weight of a branch, or *branch capacity*,[1] indicates the maximum possible rate of flow along the link. Since all these systems have a common representation, the results that can be derived from the graph will apply to any of them. The communication net, however, will be used here as a convenient example for discussion because of its important position in modern science and engineering.

- *A. The branch capacity matrix and the terminal capacity matrix*

The properties of a communication net may be described in terms of two matrices, namely, the *branch capacity matrix* and the *terminal capacity matrix*. The branch capacity matrix is a symmetrical square matrix, whose order n is the number of nodes in the net. Any term of the matrix, $b_{ij}(i \neq j)$, denotes the capacity of the branch which is connected directly between vertices i and j. Since the branch capacities b_{ii} are not defined, we shall simply write a d in the place of b_{ii} for any i. The branch capacity matrix of the communication net in Fig. 1 is, accordingly,

$$B=\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}=\begin{bmatrix} d & 1 & 0 & 0 & 5 \\ 1 & d & 2 & 2 & 0 \\ 0 & 2 & d & 3 & 1 \\ 0 & 2 & 3 & d & 2 \\ 5 & 0 & 1 & 2 & d \end{bmatrix}. \quad (1)$$

The terminal capacity matrix is also a symmetrical square matrix of order n. Each entry t_{ij} of the terminal capacity matrix is the maximum possible communication capacity between nodes i and j. As we will show in Section 2, the terminal capacity matrix of the communication net in Fig. 1 is

$$T=\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} \end{bmatrix}=\begin{bmatrix} d & 4 & 4 & 4 & 6 \\ 4 & d & 5 & 5 & 4 \\ 4 & 5 & d & 6 & 4 \\ 4 & 5 & 6 & d & 4 \\ 6 & 4 & 4 & 4 & d \end{bmatrix}. \quad (2)$$

Since for a communication net, the branch capacity matrix gives a mathematical description of its physical structure while the terminal capacity matrix offers some

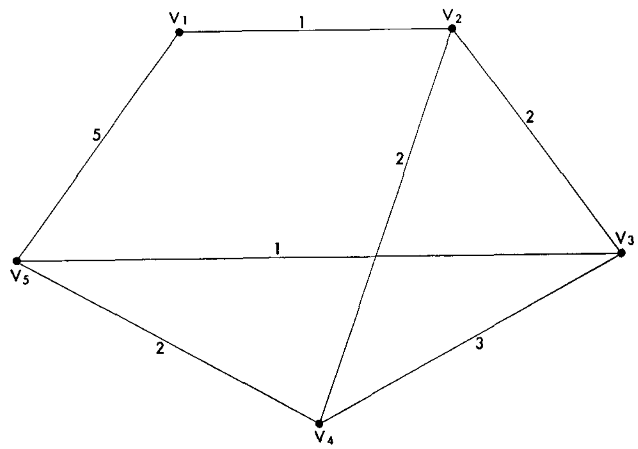


Figure 1 **A communication net.**

measure of its utility, the problem of analysis is essentially that of finding the corresponding terminal capacity matrix for a given branch capacity matrix. The problem of synthesis is that of finding the corresponding branch capacity matrix from a given terminal capacity matrix, as shown in Fig. 2. The synthesis procedure does not result in a unique net. Usually many different nets can be obtained from a single terminal capacity matrix. The terminal capacity matrix is unique, however, for a given branch capacity matrix.

• B. *Three problems of interest*

Specification of a communication net is conveniently given in terms of a terminal capacity matrix. It is reasonable to suspect that not every symmetrical matrix can be a terminal capacity matrix. The study of necessary and sufficient conditions for a symmetrical square matrix to be a terminal capacity matrix will help to decide what kind of matrix is realizable. One answer to this question was found recently by Mayeda.[2] It will be discussed briefly in Section 2A.

Once we know a certain matrix is realizable as a terminal capacity matrix, a natural question to ask is "How do we find a corresponding branch capacity matrix in a systematic manner such that the required total branch capacity is as low as possible?" Theoretically, if the matrix is realizable, there is always a realization with *minimum* branch capacity. But without a systematic method, the labor involved in the necessary search may be insurmountable. A systematic synthesis method involving straightforward procedures is described in Section 3. It is shown that the total branch capacity required with this method is a minimum. Further discussions on the synthesis method, including comments on some interesting properties of the realization, are given in Section 4.

Another interesting problem is one in which the terminal capacity requirements of the system are not available. Such is the case when we consider a military warning system, a system of interconnected canals and reservoirs for flood control, or a standby distribution system for reserve electric power supply. In many such cases the terminal capacities are either not known or will vary from time to time. The efficiency of a system of such a nature is best measured by the terminal index defined[3] as

$$I=\sum_{i<j} t_{ij}\,. \tag{3}$$

The problem is to obtain a terminal capacity matrix with maximum index where the total branch capacity and the order of the matrix are fixed. It is found that when the synthesis method in Section 3 is used, the index of any matrix may be increased to the maximum. This maximum occurs when all t_{ij} are equal. Details of these facts are discussed in Section 5.

No special mathematical background is required to follow the proofs for the theorems in this paper. The casual reader, however, may find it more interesting by simply skipping all proofs.

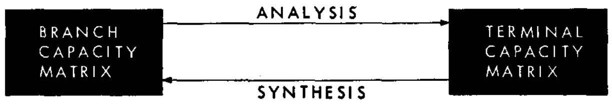


Figure 2 **Relationship between the branch capacity matrix and the terminal capacity matrix.**

2. Some properties of the communication net

• A. *Relationships between the branch capacity matrix and the terminal capacity matrix*

The notion of a *cut-set* of a graph is useful in studying the relationships between the branch capacity matrix and the terminal capacity matrix. A cut-set S_i,[4] is a collection of branches whose removal separates the graph into two disjoint non-empty subgraphs where no proper subset of S_i has this property. Since each branch in the net is assigned a positive number as its capacity, we may define a value of a cut-set as the sum of all capacities of the branches in the cut-set or

$$C_{S_k}=\sum_{b_{ij}\text{ in }S_k} b_{ij}\,, \tag{4}$$

where C_{S_k} denotes the capacity of the cut-set. If special attention is paid to a particular group of cut-sets which cut the graph is such a way that a given node i is always in one part and another given node j is always in the other part, then such a cut-set shall be denoted by the symbol $(S_k)_{i,j}$.

The terminal capacity between a pair of nodes can now be determined in terms of branch capacities as [5-7]

$$t_{ij}=\operatorname*{Min}_{S_k}\left[C_{(S_k)_{i,j}}\right]. \tag{5}$$

In other words, the terminal capacity between i and j is limited by the capacity of the cut-sets which separated nodes i and j. The cut-set which has the least capacity serves as a bottleneck which the flow of information between i and j must pass through.

In order to determine t_{ij}, we must evaluate the capacity of all cut-sets which separate i and j. For instance, the terminal capacity t_{23} of (2) is determined from the minimum of capacities of six cut-sets that separate nodes 2 and 3. The six cut-sets are shown in Fig. 3.

• B. *The necessary and sufficient condition and the partitioning process*

As is mentioned in the previous section, the determination of the terminal capacity matrix from the branch capacity matrix is unique. However, the reverse process is not only non-unique but sometimes is impossible, since some matrices cannot be realized as terminal capacity matrices. The realizability condition due to Mayeda[2] will be discussed briefly.

Since the terminal capacities t_{ij} are evaluated from the capacity of cut-sets and the value t_{ij} is the minimum, it is suggested that there must be a cut-set S_1 whose value is smallest among all cut-sets of the graph. This cut-set will cut the graph into two parts, G_A and G_B. One part will contain k nodes and the other $n-k$. The terminal capacities t_{ij} with i in G_A and j in G_B will have a value t_1, the capacity of the cut-set S_1. This suggests a partitioning process of the terminal capacity matrix. Thus, after rearrangement, T may be written in a partitioned form as

$$T=\begin{bmatrix} T_{G_A} & T_1(t_1) \\ T'_1(t_1) & T_{G_B} \end{bmatrix}, \tag{6}$$

where T_1 is a matrix whose entries are all equal to t_1 and T_{G_A} and T_{G_B} are terminal capacity matrices for G_A and G_B. Written explicitly in terms of their nodes and branches, numbering so that nodes 1 to k are in G_A and nodes $k+1$ to n are in G_B, we have

$$T=\left[\begin{array}{cccccc|cccc} t_{11} & t_{12} & \cdot & \cdot & \cdot & t_{1k} & t_1 & \cdot & \cdot & t_1 \\ t_{21} & t_{22} & \cdot & \cdot & \cdot & t_{2k} & t_1 & \cdot & \cdot & t_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t_{k1} & t_{k2} & \cdot & \cdot & \cdot & t_{kk} & t_1 & \cdot & \cdot & t_1 \\ \hline t_1 & t_1 & \cdot & \cdot & \cdot & t_1 & t_{k+1,k+1} & \cdot & \cdot & t_{k+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t_1 & t_1 & \cdot & \cdot & \cdot & t_1 & t_{n,k+1} & \cdot & \cdot & t_{n,n} \end{array}\right]. \tag{7}$$

Although (7) is more explicitly written and is very clear, (6) is preferred from the standpoint of simplicity.

The necessary and sufficient condition found by Mayeda is simply this: (1) After rearrangement,

$$T=\begin{bmatrix} T_{G_A} & T_1 \\ T'_1 & T_{G_B} \end{bmatrix}. \tag{8}$$

Where T_1 consists of t_1 only and any entry in T_{G_A} and T_{G_B} is

$$t_{ij} \geqslant t_1\,, \tag{9}$$

(2) T_{G_A} and T_{G_B} are terminal capacity matrices. Notice that if G has only two nodes then it is always a terminal capacity matrix.

If a matrix is given, the process of determination is to keep partitioning until all submatrices left are of order two.

Example: Suppose the matrix is given as

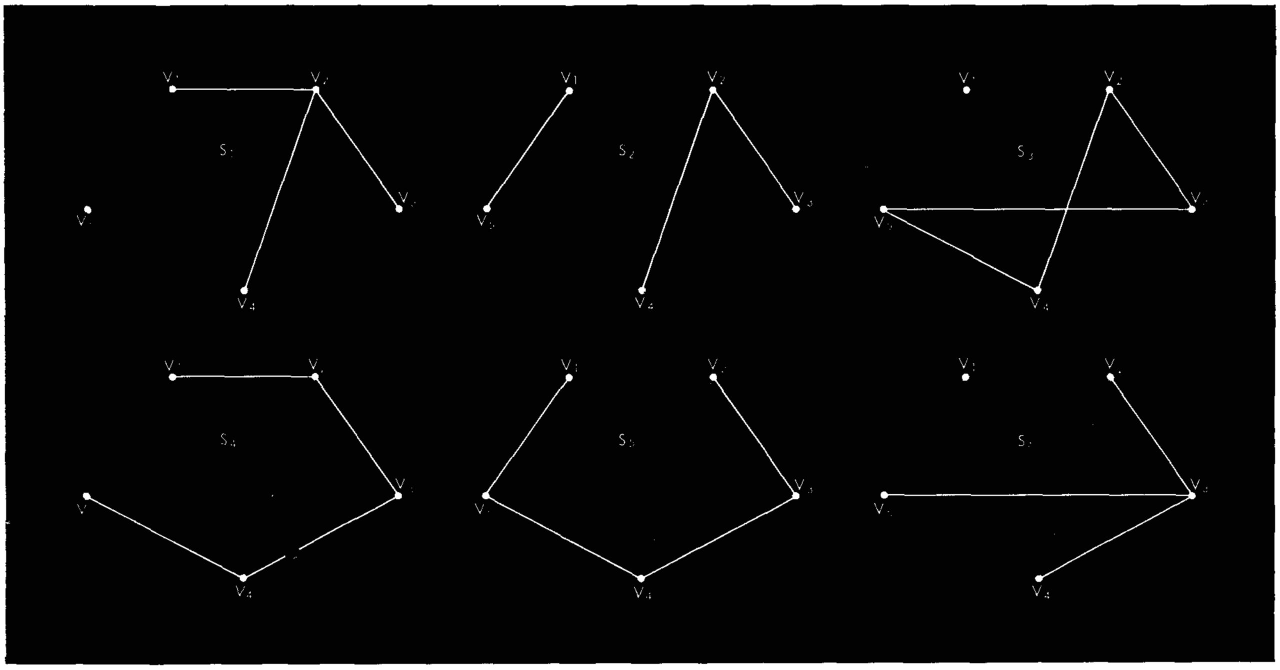
$$T=\begin{bmatrix} d & 4 & 4 & 4 & 6 \\ 4 & d & 5 & 5 & 4 \\ 4 & 5 & d & 6 & 4 \\ 4 & 5 & 6 & d & 4 \\ 6 & 4 & 4 & 4 & d \end{bmatrix}. \tag{10}$$

First it is seen that we may partition the matrix after rearranging the matrix as

$$T=\begin{bmatrix} T_{G_A} & T_1(4) \\ T'_1(4) & T_{G_B} \end{bmatrix}=\left[\begin{array}{ccc|cc} d & 5 & 5 & 4 & 4 \\ 5 & d & 6 & 4 & 4 \\ 5 & 6 & d & 4 & 4 \\ \hline 4 & 4 & 4 & d & 6 \\ 4 & 4 & 4 & 6 & d \end{array}\right], \tag{11}$$

where

Figure 3 **All cut-sets which separate nodes v_2 and v_3. Branches in cut-sets are indicated by heavy lines.**



$$T_{G_A}=\begin{bmatrix} d & 5 & 5 \\ 5 & d & 6 \\ 5 & 6 & d \end{bmatrix}, \qquad T_{G_B}=\begin{bmatrix} d & 6 \\ 6 & d \end{bmatrix}.$$

Therefore, it is seen that the matrix T is a terminal capacity matrix.

- *C. Synthesis by arbitrary choice*

The next step is to synthesize. Since we know that the minimum cut-set is 4, we write the branch capacity matrix as

$$B=\begin{bmatrix} B_{G_A} & B_1 \\ B_1' & B_{G_B} \end{bmatrix}, \tag{12}$$

and we know that

$$b_{25}+b_{21}+b_{35}+b_{31}+b_{45}+b_{41}=4\,. \tag{13}$$

We may choose arbitrarily:

$$\begin{array}{ll} b_{25}=0 & b_{21}=1 \\ b_{35}=1 & b_{31}=0 \\ b_{45}=2 & b_{41}=0 \end{array} \tag{14}$$

and then the branch capacity matrix will be

$$B=\left[\begin{array}{ccc|cc} & & & 0 & 1 \\ & B_{G_A} & & 1 & 0 \\ & & & 2 & 0 \\ \hline 0 & 1 & 2 & & \\ 1 & 0 & 0 & \multicolumn{2}{c}{B_{G_B}} \end{array}\right]. \tag{15}$$

Checking over all possible cut-sets which cut nodes 1 and 5 gives $b_{51}=5$. Again by checking all possible cut-sets which cut 2 and (3, 4), we conclude that

$$b_{23}+b_{24}=4\,. \tag{16}$$

It is seen that synthesizing this way for a large matrix, the number of cut-sets that have to be checked will increase very rapidly with the size of the matrix. Although a lot of freedom is available in distributing the branch capacity, it is not used to advantage.

In view of this, it is very desirable to have a simple and systematic method which will generate the desirable distribution for low total-branch capacity. Such a method is discussed in the next section.

3. A simple systematic method for the minimal realization of a given terminal capacity matrix

- *A. The method*

The method to be described is based on the partitioning process and can be described best along with the partitioning of the graph. Suppose the given matrix is partitioned in the form

$$T=\begin{bmatrix} T_{G_A} & T_1(t_1) \\ T_1(t_1) & T_{G_B} \end{bmatrix}. \tag{17}$$

Then its corresponding graph is also partitioned into two parts, as shown in Fig. 4. Then suppose G_A is again partitioned into G_C and G_D, and that t_1 is divided into two equal parts. The graph is shown in Fig. 5.

Now suppose the matrix is partitioned in k parts already and at the next step one of the k subgraphs will be partitioned. The situation is shown in Fig. 6, where G_k denotes the one to be partitioned next.

The process is the same, that is, each branch is divided into two equal parts, one of which is connected to G_{k_1} and the other to G_{k_2}. The branch $b_{k_1k_2}$ is determined by the difference

$$b_{k_1k_2}=t_{k_1k_2}-\tfrac{1}{2}\sum_{i=1}^{k-1} b_{ik}\,. \tag{18}$$

The partitioned graph is shown in Fig. 6b.

Since the necessary and sufficient condition is also based on the partitioning process, the ability to partition a given realizable matrix is guaranteed. The process may be written as a simple rule of thumb.

Rule: *At each partitioning process, all branches connected to the subgraph to be partitioned are divided between the two partitioned subgraphs equally. The capacity of the new branch is determined by the difference of the new terminal capacity and one-half the original branch capacity between all other subgraphs and the subgraph to be partitioned.*

- *B. Example*

Again take the terminal capacity matrix, after arrangement:

$$T=\begin{bmatrix} t_{22} & t_{23} & t_{24} & t_{25} & t_{21} \\ t_{32} & t_{33} & t_{34} & t_{35} & t_{31} \\ t_{42} & t_{43} & t_{44} & t_{45} & t_{41} \\ t_{52} & t_{53} & t_{54} & t_{55} & t_{51} \\ t_{12} & t_{13} & t_{14} & t_{15} & t_{11} \end{bmatrix} \tag{19}$$

$$=\begin{bmatrix} d & 5 & 5 & 4 & 4 \\ 5 & d & 6 & 4 & 4 \\ 5 & 6 & d & 4 & 4 \\ 4 & 4 & 4 & d & 6 \\ 4 & 4 & 4 & 6 & d \end{bmatrix}.$$

The graph is realized in successive stages as shown in Fig. 7. The total branch capacity used is

$$b_T=14\tfrac{1}{2}\,, \tag{20}$$

Figure 4 **Appearance of graph after first partition.**

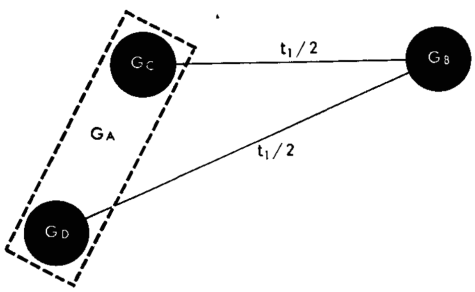


Figure 5 **Graph showing the distribution of branch capacities in the second partition.**

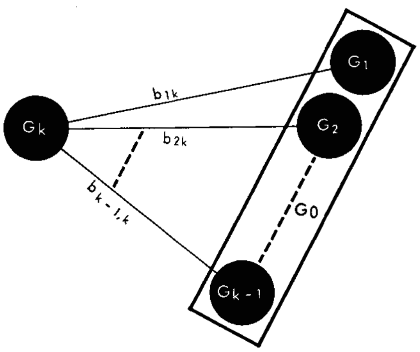


Figure 6a **Appearance of graph before k^{th} partition.**

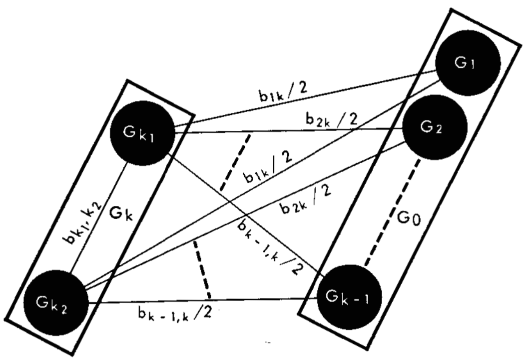


Figure 6b **Appearance of graph after k^{th} partition.**

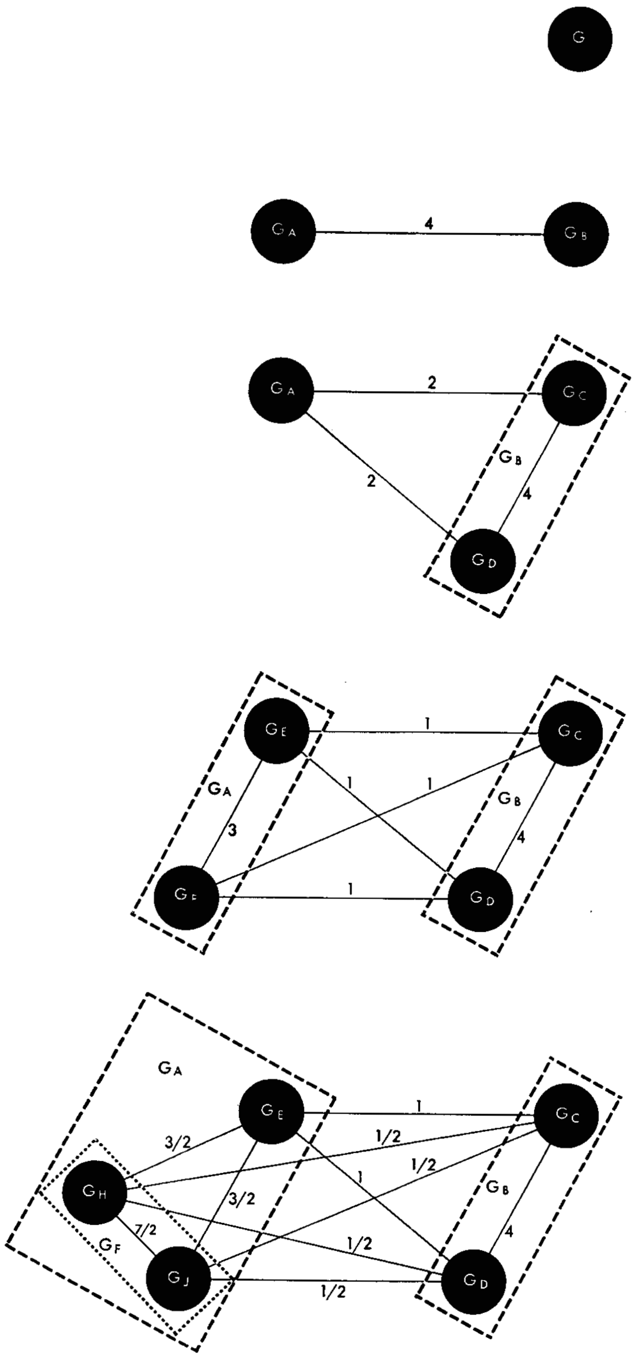


Figure 7 **An example of synthesis by successive partitions.**

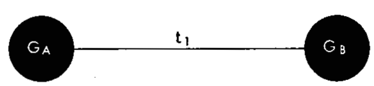


Figure 8 **Appearance of graph after first partition.**

as compared to the original value of 16 as given by the realization in Fig. 1.

- *C. The proof*

That the resultant network satisfies the given terminal capacity matrix.

At the first stage, the matrix T is partitioned as

$$T=\begin{bmatrix} T_{G_A} & T_1(t_1) \\ T'_1(t_1) & T_{G_B} \end{bmatrix}. \tag{21}$$

In this case the branch capacity matrix and the terminal capacity matrix are the same and the graph is shown in Fig. 8. As we partition again, we rearrange the matrix first so that the submatrix to be partitioned is always at the upper left corner. The terminal capacity matrix after partitioning is

$$T=\left[\begin{array}{cc|c} T_{G_C} & T_2(t_2) & \\ T'_2(t_2) & T_{G_D} & T_1(t_1) \\ \hline \multicolumn{2}{c|}{T'_1\ (t_1)} & T_{G_B} \end{array}\right]. \tag{22}$$

According to our method of synthesis, the branch matrix is given as

$$B=\begin{bmatrix} B_{G_C} & b_{CD} & b_{CB} \\ b_{CD} & B_{G_D} & b_{DB} \\ b_{CB} & b_{DB} & B_{G_B} \end{bmatrix}$$

$$=\begin{bmatrix} B_{G_C} & t_2-\dfrac{t_1}{2} & \dfrac{t_1}{2} \\ t_2-\dfrac{t_1}{2} & B_{G_D} & \dfrac{t_1}{2} \\ \dfrac{t_1}{2} & \dfrac{t_1}{2} & B_{G_B} \end{bmatrix} \tag{23}$$

where b_{IJ} denotes the sum of capacities of all branches connected between subgraphs I and J. It is seen that the cut-set that cuts G_C and G_D is equal to

$$t_2-\frac{t_1}{2}+\operatorname{Min}\left(\frac{t_1}{2},\frac{t_1}{2}\right)=t_2\,. \tag{24}$$

Then after the net has been completed up to k subgraphs, the terminal capacity matrix and the branch capacity matrix is shown as

$$T=\begin{bmatrix} T_{G_k} & T_{0k} \\ T'_{0k} & T_{G_0} \end{bmatrix} \tag{25}$$

where T_{0k} contains the terminal capacity t_{ij} for i in G_k and j in G_0. The branch capacity matrix is

$$B=\begin{bmatrix} B_{G_k} & b_{k,k-1} & b_{k,k-2} & \cdots & b_{k,1} \\ b_{k,k-1} & B_{G_{k-1}} & b_{k-1,k-2} & \cdots & b_{k-1,1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{k,1} & b_{k-1,1} & b_{k-2,2} & \cdots & B_{G_1} \end{bmatrix}. \tag{26}$$

At this stage we partition G_k into G_{k_1} and G_{k_2}, then the terminal capacity matrix will be

$$T=\left[\begin{array}{c|c|c} T_{G_{k_1}} & T_{k_1k_1} & T_{0k_1} \\ \hline T'_{k_1k_2} & T_{G_{k_2}} & T_{0k_2} \\ \hline T'_{0k_1} & T'_{0k_2} & T_{G_0} \end{array}\right], \tag{27}$$

and the branch capacity matrix will be, according to our synthesis method described in Section 3A,

$$B=\begin{bmatrix} B_{G_{k_1}} & t_{k_1,k_2}-\frac{1}{2}\sum\limits_{i=1}^{k-1} b_{ki} & \dfrac{b_{k,k-1}}{2} & \cdots & \dfrac{b_{k,1}}{2} \\ t_{k_1,k_2}-\frac{1}{2}\sum\limits_{i=1}^{k-1} b_{ki} & B_{G_{k_2}} & \dfrac{b_{k,k-1}}{2} & \cdots & \dfrac{b_{k,1}}{2} \\ \cdots & \cdots & B_{G_{k-1}} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & B_{G_1} \end{bmatrix}$$

In order to show that the given B matrix actually gives the desirable T matrix, we shall use the following argument. Consider the total of subgraph $G_1, G_2, \cdots\cdots G_{k-1}$ as parts of a graph G_0. The branch which originally connected G_k and any one of the subgraphs $G_1, G_2, \cdots\cdots G_{k-1}$ will be considered as merely connected between G_k and G_0. In order to determine the terminal capacity $t_{k_1k_2}$, the cut-set which determines this value must contain the the element $b_{k_1k_2}$. Suppose we consider three cut-sets S_1, S_2, and S_3 as shown in Fig. 9. Let G_{01} and G_{02} be the resultant subgraphs of G_0; the following relation is seen to be true, as a result of the synthesis procedure.

$$\begin{aligned} a+b&=\tfrac{1}{2}\sum_{i=1}^{k-1} b_{k,i} \\ c+d&=\tfrac{1}{2}\sum_{i=1}^{k-1} b_{k,i} \end{aligned} \tag{29}$$

also

$$\begin{aligned} a&=c \\ b&=d\,. \end{aligned} \tag{30}$$

Thus we have

$$b+c=\tfrac{1}{2}\sum_{i=1}^{k-1} b_{k,i}\,. \tag{31}$$

This result is a consequence of the synthesis method employed and is independent of how S_2 cut the graph G_0 into two parts. Therefore, we may conclude that

$$\mathrm{Min}(C_{S_1}, C_{S_3}) \leqslant C_{S_2} \tag{32}$$

and

$$b_{k_1k_2} + \tfrac{1}{2}\sum_{i=1}^{k-1} b_{k,i} = t_{k_1,k_2}\,. \tag{33}$$

Since $t_{k_1k_2} \geqslant t_{ki}$ for any i, $b_{k_1k_2}$ will always have a nonnegative value. By induction, the realization satisfies the given terminal capacity matrix. The proof that this method requires a minimum branch capacity is given in Section 4.

4. Further discusion on the synthesis method

• *A. Lower bound for total branch capacity*

With a given terminal capacity matrix it is possible to find a lower bound for the total branch capacity required.

Suppose a realization is obtained using some method of synthesis and for any node i the branch capacity of the incident set is denoted by w_i.[8] Then for any node $j(j \neq i)$ we have

$$w_i \geqslant t_{ij} \tag{34}$$

from the definition of a terminal capacity. In particular

$$w_i \geqslant t_{iz_i}\,, \tag{35}$$

where

$$t_{iz_i} = \mathrm{Max}(t_{ij}) \qquad (i \text{ fixed})\,. \tag{36}$$

If we sum over the branch capacities of all n incident sets, then

$$\sum_{i=1}^{n} w_i \geqslant \sum_{i=1}^{n} t_{iz_i}\,. \tag{37}$$

However, the left side of the above equation is seen to be equal to twice the total branch capacity, simply because each branch is connected between two nodes.

Therefore,

$$2b_T = \sum_{i=1}^{n} w_i \geqslant \sum_{i=1}^{n} t_{iz_i} \tag{38}$$

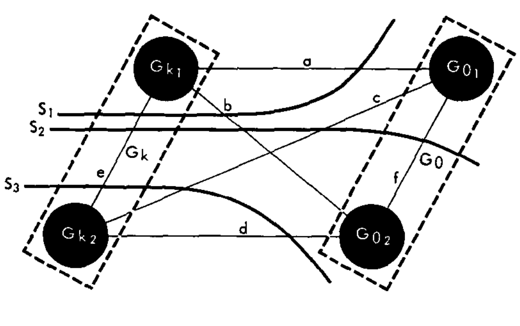


Figure 9 **Possible cut-sets at k^{th} partition. Branches in cut-sets are indicated by heavy lines.**

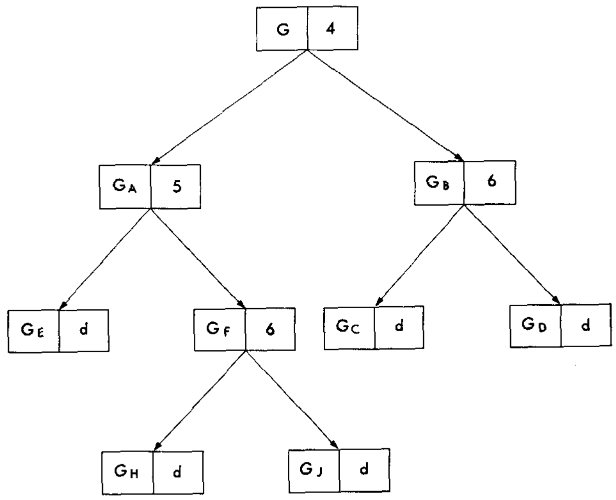


Figure 10 **Tree representation of graph partitioning in Fig. 7.**

or

$$b_T \geqslant \tfrac{1}{2}\sum_{i=1}^{n} t_{iz_i}\,. \tag{39}$$

• *B. The tree representation of the partitioning process*

Since the terminal capacity matrix contains all information needed to determine a net and the resulting net is unique if the method in Section 3 is employed, it is possible to determine the required total branch capacity without actually carrying out the synthesis procedure. This process is very simple to follow. But before going into the details of such a process it is necessary to describe first a tree representation of the breakdown of the graph. This representation is useful also in the discussion in Section 5.

The tree representation is as follows: Each subgraph will be represented by a box with G in it. On the right in each box will be the terminal capacity between the two subgraphs resulting from the partition. The partitioning of a subgraph into two is represented by drawing two lines from the original box to two new boxes. The graph in Fig. 7 for example, will have the tree representation in Fig. 10.

In general, the representation of a particular graph with n nodes will be a part of the complete tree of order n. The tree consisting of one node only is considered as of order *one*. Binary subscripts are used to keep track of the history of any subgraph. The two partitions of any subgraph G_x will be denoted as G_{x0} and G_{x1}.

• *C. Determination of the total branch capacity*

The tree representation is especially useful in the determination of total branch capacity for the realization of a given terminal capacity matrix.

In the tree representation the branch capacity that is added at each partitioning may be written on the line joining the two subgraphs of any partition. If only two steps are included, any subgraph of more than one node will have a breakdown pattern like the ones shown in Fig. 11.

The sum of the branch capaciites at the lowest level in Case I is

$$t_{x0}-\tfrac{1}{2}t_x\,. \tag{40}$$

The same sum for Case II is

$$t_{x0}-\tfrac{1}{2}t_x+t_{x1}-\tfrac{1}{2}t_x=t_{x0}+t_{x1}-t_x\,. \tag{41}$$

Notice that as we sum on the next higher level there will be a term t_x always. So, as we sum over all branch capacities the sum will be

$$b_T=\tfrac{1}{2}\sum_{\substack{G_{x0}\\ \text{is a node}}} t_x+\tfrac{1}{2}\sum_{\substack{G_{x1}\\ \text{is a node}}} t_x\,. \tag{42}$$

For instance, the terminal capacity matrix

$$T=\begin{bmatrix} d & 5 & 5 & 4 & 4\\ 5 & d & 6 & 4 & 4\\ 5 & 6 & d & 4 & 4\\ 4 & 4 & 4 & d & 6\\ 4 & 4 & 4 & 6 & d \end{bmatrix} \tag{43}$$

will require a total branch capacity of

$$b_T=6+6+\frac{5}{2}=14\,\frac{1}{2}\,. \tag{44}$$

Another way to express the Eq. (42) is

$$b_T=\tfrac{1}{2}\sum_{i=1}^{n} t_{iz_i}\,. \tag{45}$$

From the discussion in Section 4A it is seen that the required total branch capacity is a minimum if the method in Section 3 is employed.

- *D. A bonus theorem*

The total branch capacity is only dependent on the highest terminal capacity at each node. One may use this fact to advantage by increasing communication capacity at no increase of total branch capacity. This will be stated as the following theorem.

Theorem: Suppose a terminal capacity matrix is partitioned into

$$\begin{bmatrix} A & T_1\\ T_1' & B \end{bmatrix} \tag{46}$$

$$t_{ij}<\mathrm{Min}(a_{ij},\, b_{ij})\,, \tag{47}$$

and if A and B are both submatrices of order more than one, then the value t_{ij} may be made equal to

$$\mathrm{Min}(a_{ij},\, b_{ij}) \tag{48}$$

without increasing the total branch capacity required for realizing the matrix.

The terminal matrix in (43) may be improved to the following

$$T=\begin{bmatrix} d & 5 & 5 & 5 & 5\\ 5 & d & 6 & 5 & 5\\ 5 & 6 & d & 5 & 5\\ 5 & 5 & 5 & d & 6\\ 5 & 5 & 5 & 6 & d \end{bmatrix}, \tag{49}$$

of which the realization is shown in Fig. 12. The total branch capacity is seen to remain at the same value.

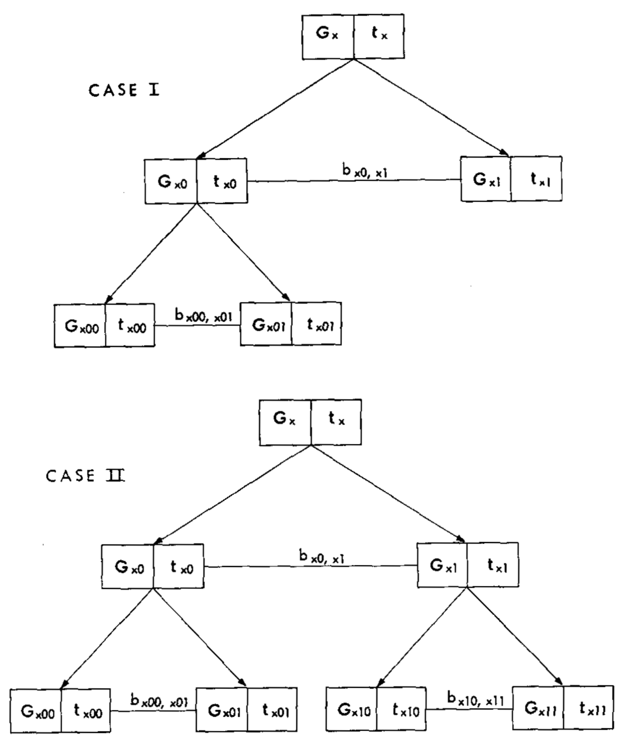
5. Nets with indeterminate terminal matrix

- *A. Optimum terminal capacity matrix for given total branch capacity*

As is discussed in Section 1, there are many cases where the terminal capacity matrix is not known. The index of communication of a net as defined in Section 1 serves as a measure of the over-all utility of the communication net. Under this criterion, we will try to find the terminal capacity matrix that will give the maximum index for a given total branch capacity. We shall denote the total branch capacity by b_T and the number of nodes of the net by n.

If we use the method in Section 3, the required total branch capacity can be determined from the terminal

Figure 11 **Two possible structures for subtrees considered in Section 5C.**



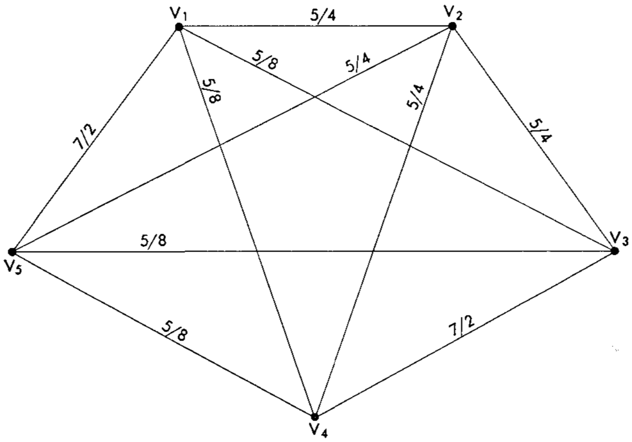


Figure 12 **Communication net for terminal capacity matrix in Eq. (49).**

capacity matrix directly. By smoothing out the differences in these terminal capacities, the index of communication can be increased steadily. The maximum point of such an increase is obtained where the terminal capacities are all made equal. The proof of this statement is given in detail in Section 5B. We see that when all terminal capacities are equal, the total branch capacities can be expressed as

$$b_T=n\frac{t}{2}\text{, or } t=2b_T/n\,. \tag{50}$$

Suppose we consider the net with equal capacity on each branch of a maximumly connected net, the terminal capacities and the total branch capacity are related as

$$\begin{aligned} b_T&=b\frac{n(n-1)}{2}\\ t&=(n-1)b\\ b_T&=\frac{n}{2}t\,. \end{aligned} \tag{51}$$

Thus it is seen that with the same total branch capacity the maximum index is the same as that of a uniformly distributed network with maximum connection. Thus for a given total branch capacity, one can expect an average communication capacity of $2b_T/n$ for a net with n nodes.

• *B. The proof*

If the terminal capacity matrix is given, it is possible to find a tree representation for the given terminal matrix. The total branch capacity for the realization is given as

$$b_T=\tfrac{1}{2}\sum_{\substack{G_{x0}\\ \text{is a node}}} t_x+\tfrac{1}{2}\sum_{\substack{G_{x1}\\ \text{is a node}}} t_x\,. \tag{52}$$

Suppose we consider a subgraph G which is partitioned into G_A and G_B as in Fig. 13. It is seen that when G contains two nodes both G_A and G_B will contribute $\frac{1}{2}t$ to the sum b_T. And the index of this subgraph is

$I=t$ and all t are equal.

Suppose G_A and G_B are both subgraphs with at least one node, namely, G_A has u nodes and G_B has v nodes, also assume in G_A and G_B all t are the same

$$\begin{aligned} t&=t_A \quad \text{in } G_A\\ t&=t_B \quad \text{in } G_B\,. \end{aligned} \tag{53}$$

Then the index of this subgraph is

$$I=t_A\cdot C_2^u+t_B\cdot C_2^v+tuv\,. \tag{54}$$

Assume $t_A\leqslant t_B$ with no loss of generality, the index can be increased by smoothing. We see that the total branch capacity which is contributed by this subgraph is

$$b_T=u\cdot\frac{t_A}{2}+v\frac{t_B}{2} \tag{55}$$

so

$$vt_B=2b_T-ut_A\,. \tag{56}$$

Substituting this into (54)

$$\begin{aligned} I&=\frac{u(u-1)}{2}t_A+\frac{(v-1)}{2}(2b_T-ut_A)+t_Auv=(v-1)b_T\\ &\quad+t_A\left[uv+\frac{u(u-1)}{2}-\frac{u(v-1)}{2}\right]\\ &=(v-1)b_T+\frac{ut_A}{2}(u+v)\,. \end{aligned} \tag{57}$$

This means if we increase $t_A(t_A\leqslant t_B)$, the index will always increase with the total branch capacity held constant. The highest value for t_A is

$$\frac{2b_T}{u+v}\,. \tag{58}$$

Substituting this in the expansion, we have, as a result,

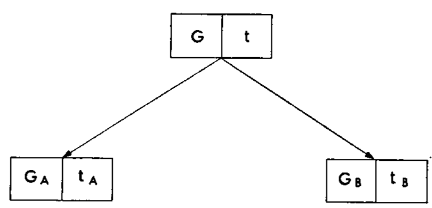
$$\begin{aligned} I&=(v-1)b_T+\frac{u(v+u)}{2}\cdot\frac{2b_T}{u+v}\\ &=b_T(v-1+u)\\ &=b_T(u+v-1)\,. \end{aligned} \tag{59}$$

By induction, then, the over-all index can be made equal to

$$I=b_T(n-1)=\frac{n(n-1)}{2}t \tag{60}$$

when all t are made equal.

Figure 13 **Tree representation for first partition of a graph.**



Since the synthesis method employed requires a minimum total branch capacity, the index as shown in (60) is the maximum for all nets with the same total branch capacity.

6. Concluding remarks

This paper has shown that a systematic synthesis procedure is possible if one treats all branches at the partition equally. A particularly simple method is obtained by making the distribution uniform. It is seen that with this method the total branch capacity required is a minimum for a given terminal capacity matrix and can be written down without actually carrying out the synthesis procedure. The total branch capacity is a sum to which each node contributes a term equal to half the value of the maximum terminal capacity associated with this particular node.

It is also shown that the index of a communication net can be increased by smoothing out the differences in terminal capacities. The maximum of index is obtained when all terminal capacities are equal.

Throughout the paper it was assumed that the branch capacities are finite. Actually this assumption can be removed if we consider a new graph where each set of nodes with infinite capacities connecting each other are considered as one node. The new graph will be a net with only finite branch capacities.

Acknowledgement

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References and Footnotes

1. Throughout this paper, the branch capacity is assumed to be a finite, non-negative real number.
2. Mayeda, W., "Terminal and Branch Capacity Matrices of a Communication Net." To appear in *IRE Transactions PGCT*.
3. This definition of the index is due to W. Mayeda.
4. The cut-set defined here is synonymous with the term *simple cut-set* in graph theory.
5. Elias, P., Feinstein, A., and Shannon, C. E., "A Note on the Maximum Flow Through a Network," *IRE Transactions on Information Theory,* **IT-2,** 117-119 (1956).
6. Ford, L. R., and Fulkerson, D. R., "Maximal Flow Through a Network," *Canadian Journal of Mathematics,* **8,** 399-404 (1956).
7. Dantzig, G. B., and Fulkerson, D. R., "On the Min-Cut Max-Flow Theorem in Networks," *Annals of Mathematical Studies,* **38,** 215-222 (1956).
8. An *incident set* is the set of branches connected to a given node.

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