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The Wave Equation in a Medium in Motion

Abstract: A model for the transverse vibrations of a tape moving between a pair of pulleys is devised using a variational procedure. It is shown by means of energy-type integrals that the energy of that portion of the tape between the pulleys is not conserved, but that there is a periodic transfer of energy into and out of the system. The solution for the wave equation is then constructed by a method which makes use of functional equations. The solution is observed to be periodic in time, and a modal decomposition of it is derived. A solution is also derived for the case of forced vibrations at the pulleys, and a class of forcing vibrations which cause unbounded solutions as time increases is isolated. In an appendix, a boundary layer effect is considered which occurs when the velocity of the tape through the pulleys approaches the sound speed of the tape.

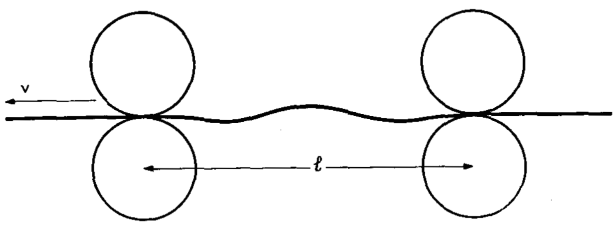
1. Introduction

In this paper* we will consider the transverse vibrations of a tape which is moving between a pair of pulleys a distance l apart. We consider the tape to be represented by an elastic string of given linear density and have in mind the situation indicated in Fig. 1. The tape (string) is drawn through the pulleys with velocity v. We wish to determine the motion of the portion of the tape between the pulleys. In particular we inquire into the effect which v has upon this motion.

In Section 2 we show how to derive the equations describing the transverse vibrations of the tape. The method for doing this makes use of Hamilton's principle, which is a classical variational technique.[1] These equations are not new. Because of the interest in the physical situation which they model, however, and because they are not especially accessible, we include the brief derivation mentioned. In Section 3 we show that there is a continued transfer of energy from that portion of the tape between the pulleys and the tape outside of them. This energy transfer implies that the vibrations of a tape which is under motion can have larger amplitudes than the vibrations of a fixed segment of tape.

In Section 4 we show how the problem can be solved. We do this by deriving formulas for extending the definition of the initial values of the tape motion. The solution is then exhibited as a simple linear functional of these extended initial values. The procedure thus produces an algorithm with which the solution may be explicitly computed. In Section 5 we show how the methods of Section 4 enable one to write down a modal decomposition of the solution. The modal representation shows that the motion of the tape is *periodic in time* and furnishes a second procedure for computing the solution to the problem.

Figure 1



In Section 6 we consider the case of forced vibrations introduced at the pulleys and show how to solve the problem in this case. From our solution of the problem in this case, we are able to conclude that a large class of forcing motions can lead to unbounded solutions. Our solution in this case demonstrates explicitly the way in which the solution becomes large. In particular, by specializing the end conditions to be harmonic, we are able to obtain the resonant frequencies of our problem. These resonant frequencies must be considered in the design of the pulley system. We also consider the passage of a splice through the system and point out the possibilities of a splice causing resonance.

In the Appendix we consider the situation when the velocity of the tape approaches the sound speed of the tape. We show that a boundary-layer phenomenon occurs in such an event and indicate how the solution may be approximated in this case.

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2. The equation of motion

- *Derivation*

To obtain the equation of motion, we will use Hamilton's principle, which states that the variation of the time integral of the kinetic minus the potential energy of a segment at the tape is zero. See Fig. 2.

If $u(x, t)$ is the transverse displacement of the tape at position x at time t, then the velocity of the segment of tape at position x and time t is $Du(x, t)/Dt$ or

$$u_t + x_t u_x = u_t - v u_x \, . \tag{2.1}$$

Thus the kinetic energy of the segment of tape between x_1 and x_2 is

$$K = \int_{x_1}^{x_2} \frac{1}{2} \rho [v^2 + (u_t - v u_x)^2] dx \, , \tag{2.2}$$

where ρ is the linear density of the tape.

The potential energy of this segment is

$$V = \int_{x_1}^{x_2} \frac{1}{2} q u_x^2 dx \, , \tag{2.3}$$

where q is the tension. Hamilton's principle now states

$$\delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} \{\rho[v^2 + (u_t - v u_x)^2] - q u_x^2\} dx dt = 0 \, . \tag{2.4}$$

Applying the Euler operator

$$\frac{\partial}{\partial t} \frac{\partial}{\partial u_t} + \frac{\partial}{\partial x} \frac{\partial}{\partial u_x} - \frac{\partial}{\partial u} \tag{2.5}$$

to the integrand in (2.4) gives us the equation of motion:[2]

$$u_{tt} - 2v u_{xt} + (v^2 - c^2) u_{xx} = 0 \, , \tag{2.6}$$

if v is a constant, or

$$u_{tt} - 2v u_{xt} + (v^2 - c^2) u_{xx} - \dot{v} u_x = 0 \tag{2.7}$$

if v is a non-constant function of time. Here $c = (q/\rho)^{\frac{1}{2}}$ is the sound speed. A dot represents time differentiation. To the equation of motion, we append the boundary conditions

$$u(0, t) = u(l, t) = 0 \, , \tag{2.8}$$

which represent the fact that there is no transverse motion of the tape at the pulleys.

- *A transformation*

If we make the transformation

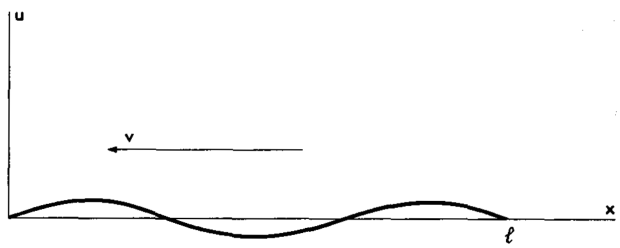
$$x + \int_0^t v(\sigma) d\sigma = \xi \, , \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \, ,$$
$$t = \tau \, , \qquad \frac{\partial}{\partial t} = v \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} \, , \tag{2.9}$$

the equation of motion becomes

$$u_{\tau\tau} - c^2 u_{\xi\xi} = 0 \, , \tag{2.10}$$

while the boundary conditions become

Figure 2



$$u(v\tau, \tau) = u(l + v\tau, \tau) = 0 \, , \tag{2.11}$$

for v constant, and

$$u\left(\int_0^\tau v(\sigma) d\sigma, \tau\right) = u\left(l + \int_0^\tau v(\sigma) d\sigma, \tau\right) = 0 \, , \tag{2.12}$$

for v a non-constant function of time.

3. Energy considerations

- *Energy*

From Section 2 we may write the energy of the tape (between the pulleys) as

$$W = \frac{1}{2} \int_0^l \{\rho[v^2 + (u_t - v u_x)^2] + q u_x^2\} dx \, . \tag{3.1}$$

Thus

$$\frac{dW}{dt} = \frac{1}{2} \rho \frac{d}{dt} \int_0^l [u_t^2 + (c^2 - v^2) u_x^2 - 2v u_x u_t] dx \, ,$$
$$= \rho \int_0^l [u_t u_{tt} + (c^2 - v^2) u_x u_{xt} - v u_t u_{xt} - v u_x u_{tt}] dx \, . \tag{3.2}$$

Using the equation of motion, (3.2) can be written as

$$\frac{dW}{dt} = \rho \int_0^l \{[u_t - v u_x][2v u_{xt} + (c^2 - v^2) u_{xx}] + [c^2 - v^2] u_x u_{xt}\} dx \, . \tag{3.3}$$

Equation (3.3) can by integration by parts be brought into the form

$$\frac{dW}{dt} = \frac{\rho}{2} \{v[v^2 - c^2][u_x^2(l, t) - u_x^2(0, t)] - 2v^2 \frac{d}{dt} \int_0^l u_x^2 dx\} \, . \tag{3.4}$$

This expression is not in general zero. This may be seen in the following way. By adjusting the initial values $u(x, 0) = h(x)$, $u_t(x, 0) = k(x)$, $u_x(x, 0) = h'(x)$, and $u_{xt} = k'(x)$, the expression for dW/dt at $t = 0$ may be made non-zero. We may also arrange that dW/dt be continuous so that dW/dt is not zero for small t. We will see below that u is periodic. Thus dW/dt is periodic and then dW/dt is non-zero in general.

From this we may conclude that the energy of the portion of the tape between the pulleys is not conserved; i.e., there is a continual and periodic transfer of energy from the system outside the pulleys to that part of the tape between the pulleys. Thus we see that the amplitude of the motion with $v \neq 0$ can under certain conditions be very much larger than the amplitude when v is zero for the same initial conditions.

• *Uniqueness*

The uniqueness of the solution to this problem is well known. We merely remark that various integrations and use of the equation of motion enable the following identity to be derived:

$$\frac{d}{dt}\int_0^l [u_t^2+(c^2-v^2)u_x^2]dx=0\,. \tag{3.5}$$

This identity implies the uniqueness of the problem when $c>v$, according to a well-known argument.

4. The solution of the equation of motion

In this section we will show how to write down the solution to the problem using the decomposability of the solution into waves traveling in two directions. We will use the equation written in (ξ, τ) coordinates.

• *Constant v*

We write

$$u(\xi, \tau)=f(\xi+c\tau)+g(\xi-c\tau)\,. \tag{4.1}$$

The first relation in (2.11) gives

$$f(a\tau)=-g(b\tau),\ a=v+c,\ b=v-c \tag{4.2}$$

or

$$g(\tau)=-f\left(\frac{a\tau}{b}\right). \tag{4.3}$$

Inserting this into (4.1) gives

$$u(\xi, \tau)=f(\xi+c\tau)-f\left(\frac{a}{b}(\xi-c\tau)\right). \tag{4.4}$$

The section relation in (2.11) inserted into (4.4) gives a periodicity condition on $f(t)$, viz.,

$$f(a\tau+l)=f\left(\frac{a}{b}(b\tau+l)\right) \tag{4.5}$$

or

$$f(\tau)=f(\tau+\alpha l),\ \text{with}\ \alpha=\frac{2c}{c-v}\,. \tag{4.6}$$

Let $h(\xi)$ and $k(\xi)$ be respectively the initial value and initial velocity of $u(\xi, \tau)$. Then from (4.4) we have

$$h(\xi)=f(\xi)-f\left(\frac{a\xi}{b}\right) \tag{4.7}$$

and

$$\frac{k(\xi)}{c}=f'(\xi)+\frac{a}{b}f'\left(\frac{a}{b}\xi\right). \tag{4.8}$$

If we let

$$K(\xi)=\int_0^\xi \frac{k(\sigma)}{c}d\sigma\,, \tag{4.9}$$

then (4.8) yields

$$K(\xi)=f(\xi)+f\left(\frac{a}{b}\xi\right)-2f(0)\,. \tag{4.10}$$

By adding (4.7) and (4.10), and then subtracting them, we obtain two expressions for $f(\xi)-f(0)$. Equating these two expressions for $f(\xi)$ gives

$$h(\xi)+K(\xi)=-\mathrm{h}\left(\frac{b}{a}\xi\right)+K\left(\frac{b}{a}\xi\right). \tag{4.11}$$

This is not a relation between the initial data because $b<0$, and $h(\xi)$ and $k(\xi)$ are given only for $0\leqslant\xi\leqslant l$.

Applying the periodicity condition (4.6) to either member of (4.11) gives

$$h(\xi)+K(\xi)=h(\xi+\alpha l)+K(\xi+\alpha l)\,. \tag{4.12}$$

This is not a relation between the initial data because $\alpha>2$. If we let $F=h+K$, $G=-h+K$, and $\beta=b/a$, we get from (4.11) and (4.12)

$$F(\xi)=F(\xi+\alpha l)+G(\beta\xi)\,. \tag{4.13}$$

The question of existence is thus reduced to extending the initial data $h(\xi)$ and $k(\xi)$ so that (4.13) is satisfied for all ξ. If this is done, the solution is from (4.4).

$$u(\xi, \tau)=\frac{1}{2}\left[h(\xi+c\tau)+K(\xi+c\tau)-h\left(\frac{a}{b}(\xi-c\tau)\right)\right.$$
$$\left.-K\left(\frac{a}{b}(\xi-c\tau)\right)\right]. \tag{4.14}$$

To show that we can extend $h(\xi)$ and $k(\xi)$ so that (4.13) is satisfied for all ξ, we proceed as follows. $F(\xi)$ and $G(\xi)$ are known for ξ in $(0,l)$. As ξ varies in $(0,l/\beta)$, $\beta\xi$ varies in $(0,l)$. Thus as the argument of G (in (4.13)) varies in $(0,l)$ (where G is known), the argument of F (in (4.13)) varies in $(0,l/\beta)$. Thus we know F on $(l/\beta, l)$, (recall $\beta=(b/a)<0$).

Now the length of the interval $(l/\beta, l)$ is

$$l\left(1-\frac{1}{\beta}\right)=l\left(\frac{b-a}{b}\right)=\alpha l\,. \tag{4.15}$$

Thus since F has period αl, we know F everywhere, and by (4.13) we know G everywhere; i.e., the initial data can indeed be appropriately extended. Using (4.1) and (4.13) and implementing the extension argument, a procedure for computing $u(x, t)$ is at hand.

• *Variable $v(t)$*

We proceed as in the case of constant v. Then let

$$u(\xi, \tau)=f(\xi+c\tau)+g(\xi-c\tau)\,. \tag{4.16}$$

The first relation in (2.12) gives

$$f[a(\tau)\tau]=-g[b(\tau)\tau]\ ,$$

$$a(\tau)=\frac{1}{\tau}\int_0^\tau v(\sigma)d\sigma+c\ , \tag{4.17}$$

$$b(\tau)=\frac{1}{\tau}\int_0^\tau v(\sigma)d\sigma-c\ .$$

Let

$$b(\tau)\tau=t\ , \tag{4.18}$$

and let

$$\tau=w(t) \tag{4.19}$$

be its inverse. Then (4.17) becomes

$$\begin{aligned} g(t)&=-f\{a[w(t)]w(t)\}\\ &=-f[t+2cw(t)]\ . \end{aligned} \tag{4.20}$$

Thus

$$u(\xi,\tau)=f(\xi+c\tau)-f[\xi-c\tau+2cw(\xi-c\tau)]\ . \tag{4.21}$$

Now the second relation in (2.12) gives

$$f[a(\tau)\tau+l]=f\{b(\tau)\tau+l+2cw[b(\tau)\tau+l]\}.. \tag{4.22}$$

This implies

$$f[t+2cw(t)+l]=f[t+l+2ctw(t+l)]\ , \tag{4.23}$$

which may be written as

$$f[x+2cw(x-l)]=f[x+2cw(x)]\ . \tag{4.24}$$

This is the analogue of the periodicity condition in (4.6).

Now continuing exactly as in the case of constant v, we may obtain our solution provided the following analogue of (4.13) can be used to extend our initial data, viz.,

$$F(\xi)=G[\lambda(\xi)]=F\{\lambda(\xi)+2cw[\lambda(\xi)-l]\}\ . \tag{4.25}$$

Here $\lambda(x)$ is the inverse function of $y+cw(y)$.

It is not clear when h and k can be extended in accordance with (4.25). However, we may note that when this is possible, the values of h and k at the extended arguments are those of h and k on the interval $(0, l)$. Thus at any point (ξ, τ), the solution $u(\xi, \tau)$ is the usual linear combination of some of the values of h and k on $(0, l)$.

5. Modal representation (constant v)

From the periodicity condition (4.6) for $f(\xi)$, we have

$$f(\xi)=\sum_{-\infty}^{\infty} f_n \exp\left(\frac{2n\pi i\xi}{\alpha l}\right)$$

$$f_n=\frac{1}{\alpha l}\int_0^{\alpha l} f(\xi)\exp\left(-\frac{2n\pi i\xi}{\alpha l}\right)d\xi\ . \tag{5.1}$$

To determine f_n, the knowledge of $f(\xi)$ or $h+K$ on an interval larger than $(0, l)$ is required. We have

$$\begin{aligned} u(\xi,\tau)=\sum_{-\infty}^{\infty} f_n\Bigg[&\exp\left(\frac{2n\pi i}{\alpha l}(\xi+c\tau)\right)\\ &-\exp\left(\frac{2n\pi i}{\alpha l}(\xi-c\tau)\right)\Bigg]\ . \end{aligned} \tag{5.2}$$

If $f_n=\sigma_n+i\tau_n$, $s_n=\sigma_n+\sigma_{-n}$, and $t_n=\tau_n-\tau_{-n}$,

then (5.2) can be written as

$$\begin{aligned} u(\xi,\tau)=\sum_1^{\infty}\Bigg[&s_n\left(\cos\frac{2n\pi}{\alpha l}(\xi+c\tau)-\cos\frac{2n\pi}{(b/a)\alpha l}(\xi-c\tau)\right)\\ &-t_n\left(\sin\frac{2n\pi}{\alpha l}(\xi+c\tau)-\sin\frac{2n\pi}{(b/a)\alpha l}(\xi-c\tau)\right)\Bigg]\ . \end{aligned} \tag{5.3}$$

In the x, t coordinates, this becomes

$$\begin{aligned} u(x,t)=&\sum_1^{\infty}\Bigg[s_n\left(\cos\frac{2n\pi}{\alpha l}x-\cos\frac{2n\pi}{(b/a)\alpha l}x\right)\\ &-t_n\left(\sin\frac{2n\pi}{\alpha l}x-\sin\frac{2n\pi}{(b/a)\alpha l}x\right)\Bigg]\cos\frac{2n\pi}{\alpha l}at\\ &+\sum_1^{\infty}\Bigg[-s_n\left(\sin\frac{2n\pi}{\alpha l}x-\sin\frac{2n\pi}{(b/a)\alpha l}x\right)\\ &-t_n\left(\cos\frac{2n\pi}{\alpha l}x-\cos\frac{2n\pi}{(b/a)\alpha l}x\right)\Bigg]\sin\frac{2n\pi}{\alpha l}at\ . \end{aligned} \tag{5.4}$$

This clearly illustrates the modal behavior of the solution. It also shows that u has period $\frac{\alpha l}{a}=\frac{2cl}{c^2-v^2}$ in t. The numbers f_n, σ_n, τ_n, s_n, and t_n can readily be determined from (5.1), after the initial data is extended from $(0, l)$ to $(0, \alpha l)$ in accordance with (4.13) and the extension argument immediately following (4.13).

6. The response of the tape to forcing motions at the pulleys

In this section we will consider our tape to be subject to forced motions at its ends (the pulleys). We will show how to solve the problem in this case. We will also show that a large class of forcing motions can lead to unbounded motions of the tape. In particular we will see that certain harmonic motions of the ends are in this class. The latter situation is the phenomenon of resonance.

We will use the (ξ, τ) coordinates.

Since our problem is linear and since the solution with zero boundary conditions is periodic and bounded, we may take as initial data

$$h(\xi)=k(\xi)=0\ . \tag{6.1}$$

At the boundaries we take

$$\begin{aligned} u(v\tau,\tau)&=\phi_1(\tau)\\ u(v\tau+l,\tau)&=\phi_2(\tau)\ , \end{aligned} \tag{6.2}$$

with $\phi_1(\tau)=\phi_2(\tau)=0$ for $\tau\leqslant 0$, and with $\phi_1(\tau)$ and $\phi_2(\tau)$ continuous for all τ.[3]

Let

$$u(\xi, \tau)=f(\xi+c\tau)+g(\xi-c\tau)\,. \qquad (6.3)$$

Applying (6.2) to (6.3) we may obtain

$$f(x)=\phi_1\left(\frac{x}{a}\right)-g\left(\frac{b}{a}\,x\right), \qquad (6.4)$$

with $g(x)$ to be determined from

$$\phi_1\left(\frac{x}{b}\right)-\phi_2\left(\frac{x}{b}-\frac{l}{a}\right)$$

$$=g(x)-g\left[x+l\left(1-\frac{b}{a}\right)\right]. \qquad (6.5)$$

The solution to (6.5) is

$$g(x)=\begin{cases}0, & x>0\\ \sum_0^{Z_1}\phi_1\left(\dfrac{x+n\alpha l}{b}\right)-\sum_0^{Z_2}\phi_2\left(\dfrac{x+n\alpha l}{b}-\dfrac{l}{a}\right), & x<0\,,\end{cases} \qquad (6.6)$$

where $Z_1=\left[\dfrac{-x}{\alpha l}\right]$ and $Z_2=\left[\dfrac{b}{a\alpha}-\dfrac{x}{\alpha l}\right]$

All summations are to proceed up to the integral part of the upper limit of summation. Since $b=v-c<0$, we see that $f(x)=0$ for $x<0$.

Thus we have

$$u(\xi,\tau)=\phi_1\left(\frac{\xi+c\tau}{a}\right)-\sum_0^{Z_3}\phi_1\left(\frac{(b/a)(\xi+c\tau)+n\alpha l}{b}\right)$$

$$+\sum_0^{Z_4}\phi_2\left(\frac{(b/a)(\xi+c\tau)+n\alpha l}{b}-\frac{l}{a}\right)$$

$$+\begin{cases}\sum_0^{Z_5}\phi_1\left(\dfrac{\xi-c\tau+n\alpha l}{b}\right)-\sum_0^{Z_6}\phi_2\left(\dfrac{\xi-c\tau+n\alpha l}{b}-\dfrac{l}{a}\right), & \xi-c\tau<0\,,\\ 0, & \xi-c\tau>0\end{cases} \qquad (6.7)$$

where

$$Z_3=\left[-\frac{b}{a}\frac{\xi+c\tau}{\alpha l}\right],\qquad Z_4=\left[\frac{b}{a\alpha}-\frac{b}{a}\frac{\xi+c\tau}{\alpha l}\right],$$

$$Z_5=\left[-\frac{\xi-c\tau}{\alpha l}\right],\text{ and } Z_6=\left[\frac{b}{a\alpha}-\frac{\xi-c\tau}{\alpha l}\right].$$

Since we are interested in determining when as τ becomes large, the solution becomes large, we assume that $\tau>l/(c-v)$. This implies that $\xi-c\tau<0$, and simplifies (6.7). Suppose further that $\phi_2\equiv 0$. Then

$$u(\xi,\tau)=-\sum_1^{Z_3}\phi_1\left(\frac{(b/a)(\xi+c\tau)+n\alpha l}{b}\right)$$

$$+\sum_0^{Z_5}\phi_1\left(\frac{\xi-c\tau+n\alpha l}{b}\right), \qquad (6.8)$$

where $Z_3=\left[-\dfrac{b}{a}\dfrac{\xi+c\tau}{\alpha l}\right]$ and $Z_5=\left[-\dfrac{\xi-c\tau}{\alpha l}\right]$.

As $\tau\to\infty$, the number of terms in these sums gets larger and tends also to infinity. To illustrate the possibilities let us consider what happens at the midpoint of the tape; i.e., let $\xi=v\tau+l/2$. From (6.8) we obtain

$$u\left(v\tau+\frac{l}{2},\tau\right)=-\sum_1^{Z_7}\phi_1\left(\tau+\frac{l}{2a}+\frac{n\alpha l}{b}\right)$$

$$+\sum_0^{Z_8}\phi_1\left(\tau+\frac{l}{2b}+\frac{n\alpha l}{b}\right), \qquad (6.9)$$

where $Z_7=\left[-\dfrac{b\tau}{\alpha l}-\dfrac{b}{2a\alpha}\right]$ and $Z_8=\left[-\dfrac{b\tau}{\alpha l}-\dfrac{1}{2\alpha}\right]$.

Now the arguments of ϕ_1 in these sums (during summation) move in steps of $l[(b-a)/b^2]$ while the arguments in the two sums have a phase difference of one-half this value, $l/2[(b-a)/b^2]$. It is clear that by adjusting the sign and/or amplitude of $\phi_1(x)$ as x varies, the expression in (6.9) will tend to infinity with τ. Thus a large class of end conditions will lead to solutions which grow as large as one pleases as time approaches infinity. If in particular

$$\phi_1(\tau)=\sin\omega\tau,\quad \omega=(\pi/cl)(v-c)^2=(2\pi/l)\,\frac{b^2}{a-b}\,, \qquad (6.10)$$

then $u(\xi,\tau)$ will gradually become as large as one desires by taking τ sufficiently large. Thus $\omega=(\pi/cl)\ (v-c)^2$ is a resonant frequency of the system. Indeed we see that the free vibration frequency, $2\pi a/\alpha l$, found in Section 5, is a resonant frequency. The advantage in obtaining the resonant frequencies from an exact expression, (6.9), of the solution is that one need not be content with a statement that the solution tends to infinity with time, but can see exactly how this happens. In particular he can compute the time it takes for the solution to reach a given amplitude. This attribute could be of importance in specific design problems.

The passage of a splice

Since a splice causes a bump to exist on the piece of tape, the passage of splice through the pulleys may be interpreted by choosing $\phi_1(\tau)$ and $\phi_2(\tau)$ to be step functions which are zero then equal to the extra thickness of a splice and then zero again. If d is the length of the splice, then ϕ_1 and ϕ_2 will each be non-zero over an interval of d/v in duration. The form of the solution shows that $u(\xi,\tau)$ will be a linear combination of the added thickness of a splice with ± 1 for coefficients. Since the arguments in the sums in (6.7) are taken in increments of $|\alpha l|=2cl/(c-v)^2$, we see that the larger c is (the smaller this increment) the more non-zero terms will appear. In particular we see that the number of non-zero terms will be proportional to the time, α/v, that it takes a splice to pass through a pulley, divided by the increment of summation; i.e.,

$$\frac{(d/v)}{[2cl/(c-v)^2]} \sim \frac{dc}{2vl}$$

for large c (large tape tension).

Thus the solution will vary in some repetitive manner and having values in integral multiples of the added thickness of a splice. The maximum amplitude will be proportional to the added thickness of a splice, h, times the number of terms in the summation, that is, $hdc/2vl$ for large c.

If a succession of splices passes through the system, a situation where the solution grows very large might develop. This may be seen by examining (6.8). For although $\phi_1=0$ or h, the frequencies of splices might be just of the form that the negative (or positive) terms in (6.8) predominate in number.

Appendix: Boundary layers

In this appendix we show how the formalism of boundary-layer theory arises in our problem.

For constant v our equation is

$$u_{tt}=(c^2-v^2)u_{xx}+2vu_{xt}\,. \tag{A-1}$$

The characteristics of this equation are the straight lines whose slopes are

$$m=v\pm c\,. \tag{A-2}$$

This equation is then always hyperbolic. We note, however, that when $v\rightarrow c-$, the equation cannot satisfy the boundary value problem. For in this case, the solution of the formal limit equation is

$$U=\alpha(x)+\beta(x+2ct)\,, \tag{A-3}$$

a standing wave plus a wave moving with velocity $2c=v+c$. The boundary conditions and initial condition, $u=h(x)$, imposed only on (A-3) would already yield $U=h(x)$. Thus the solution of the formal limit equation (as $v\rightarrow c-$) cannot satisfy all the conditions of the original problem. This comes about because one family of characteristics becomes the set of straight lines with zero slope. In the theory of singular perturbations, boundary value problems are considered which, when certain formal limits are taken, become unable to satisfy all the conditions they formally did. Here we do not have a singular perturbation, but still we have a boundary-layer effect.

We anticipate on physical grounds that the solution will approach a standing wave and a wave with velocity $2c$, as in (6.3). Thus we do not want to make (A-3) into $U=h(x)$, a standing wave only. This means that we should refrain from making U satisfy the boundary conditions, and add to U a term of importance near the boundaries. This suggests the boundary-layer transformation.

In (A-1), let

$$c^2-v^2=\varepsilon(v+c)\,. \tag{A-4}$$

For the left boundary ($x=0$), let

$$x=\varepsilon s\,. \tag{A-5}$$

Letting $\phi(s,t)=\lim_{\varepsilon\rightarrow 0} u(\varepsilon s,t)$, we get from (A-1)

$$0=\phi_{ss}+\phi_{st}\,. \tag{A-6}$$

Then

$$\phi=\gamma(t)+\delta(s+t)=\gamma(t)+\delta\left(\frac{x}{\varepsilon}+t\right). \tag{A-7}$$

For the right boundary ($x=l$), let

$$l-x=\varepsilon\sigma,\ \psi=\lim_{\varepsilon\rightarrow 0} u(l-\varepsilon\sigma,t)\,. \tag{A-8}$$

Then

$$0=\psi_{\sigma\sigma}-\psi_{\sigma t}\,. \tag{A-9}$$

Hence,

$$\psi=\xi(t)+\eta(\sigma+t)=\xi(t)+\eta\left(\frac{l-x}{\varepsilon}+t\right). \tag{A-10}$$

We expect u to be approximated by $U+\phi+\psi$ as $\varepsilon\rightarrow 0$. Since we expect u to approach a standing wave plus a wave of velocity $2c$, then $\delta(t)+\xi(t)$ should tend to zero with ε. As a first approximation let us set them to zero.

Applying the initial conditions $u=h(x)$, $u_t=\chi(x)$, we get

$$\alpha(x)+\beta(x)+\delta\left(\frac{x}{\varepsilon}\right)+\eta\left(\frac{l-x}{\varepsilon}\right)=h(x)\,, \tag{A-11}$$

and

$$2c\beta'(x)+\delta'\left(\frac{x}{\varepsilon}\right)+\eta'\left(\frac{l-x}{\varepsilon}\right)=\chi(x)\,. \tag{A-12}$$

We integrate (A-12) and write the result as

$$2c\beta(x)+\varepsilon\delta\left(\frac{x}{\varepsilon}\right)-\varepsilon\eta\left(\frac{l-x}{\varepsilon}\right)=L(x). \tag{A-13}$$

Applying the boundary conditions to $U+\phi+\psi$ gives

$$\alpha(0)+\beta(2ct)+\delta(t)+\eta\left(\frac{l}{\varepsilon}+t\right)=0, \tag{A-14}$$

and

$$\alpha(l)+\beta(l+2ct)+\delta\left(\frac{l}{\varepsilon}+t\right)+\eta(t)=0. \tag{A-15}$$

Solving these two equations for $\beta(x)$ gives respectively

$$\beta(x)=-\delta\left(\frac{x}{2c}\right)-\eta\left(\frac{l}{\varepsilon}+\frac{x}{2c}\right)-\alpha(0)$$

and

$$\beta(x)=-\delta\left(\frac{l}{\varepsilon}+\frac{x-l}{2c}\right)-\eta\left(\frac{x-l}{2c}\right)-\alpha(l).$$

Equating, we get

$$\delta\left(\frac{x}{2c}\right)+\eta\left(\frac{l}{\varepsilon}+\frac{x}{2c}\right)$$

$$+\alpha(0)=\delta\left(\frac{l}{\varepsilon}+\frac{x-l}{2c}\right)+\eta\left(\frac{x-l}{2c}\right)+\alpha(l)$$

$$=-\frac{\varepsilon}{2c}\left(\frac{x}{\varepsilon}\right)+\frac{\varepsilon}{2c}\eta\left(\frac{l-x}{\varepsilon}\right)+\frac{1}{2c}L(x). \tag{A-16}$$

If we can solve these two equations for δ and η, we insert the result in (A-13) to determine β. Then (A-11) gives $\alpha(x)$.

References

1. This method is employed by F. R. Archibald and A. G. Emslie, "The Vibration of a String Having Uniform Motion Along its Length," *Journal of Applied Mechanics,* **25,** 347-348 (1958).
2. Courant and Hilbert, *Methoden der Mathematischen Physik,* Vol. I, Springer, Berlin, 1937. See p. 245 for details concerning a similar use of Hamilton's principle.
3. These assumptions are made for technical reasons. One of these reasons is that we wish to avoid discontinuous (weak) solutions of the wave equation.

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