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Theory of a Fast-Switching Electron-Beam Frequency Divider*

Abstract: A velocity-modulated electron-beam microwave tube is described which can be operated as a frequency divider. Its operation is analyzed in terms of velocity-modulation bunching theory, neglecting space-charge forces. Because of the existence of two stable states opposite in phase, such a divider can be advantageously employed in a microwave logical system. The transient behavior of the device is discussed, particularly with reference to the time required to switch the device from one of its stable states to the other. Factors involved in the minimization of this time interval are analyzed.

1. Introduction

For certain computing applications there is a need for a microwave frequency divider, which would switch from one to the other of its two stable phases in an extremely short time. Such a device, which will be referred to as the *klystron divider,* consists of a single-cavity klystron oscillator, of either the drift tube or reflex type, with an additional bunching cavity (called the *prebunching cavity*) placed between the cathode and oscillator cavity, as shown in Fig. 1. The oscillator cavity is tuned to the subharmonic frequency, henceforth assumed to be one-half the input frequency, while the prebunching cavity is tuned to and is excited by the input frequency. Such a device can be operated as a frequency divider simply by setting the beam current at a value somewhat below that of the starting current for the oscillator. In a manner to be described, the excitation of the prebunching cavity and the concomitant prebunching of the beam makes the electronic conductance of the beam at the subharmonic frequency sufficiently negative to permit the amplitude build-up of the subharmonic in the oscillator cavity.

A qualitative description of the mechanism involved is particularly simple when it is possible to neglect the velocity modulation of the prebunched beam compared to the density modulation it has developed by the time it has entered the oscillator cavity. This is, in fact, a good approximation when the prebunching drift angle is chosen to be large compared to the effective oscillator

*See Reference 1.

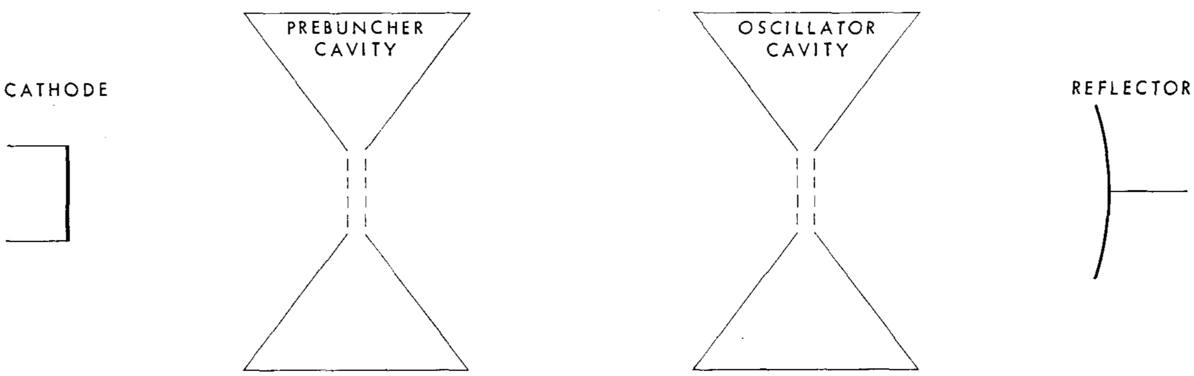


Figure 1 **Reflex-type frequency divider.**

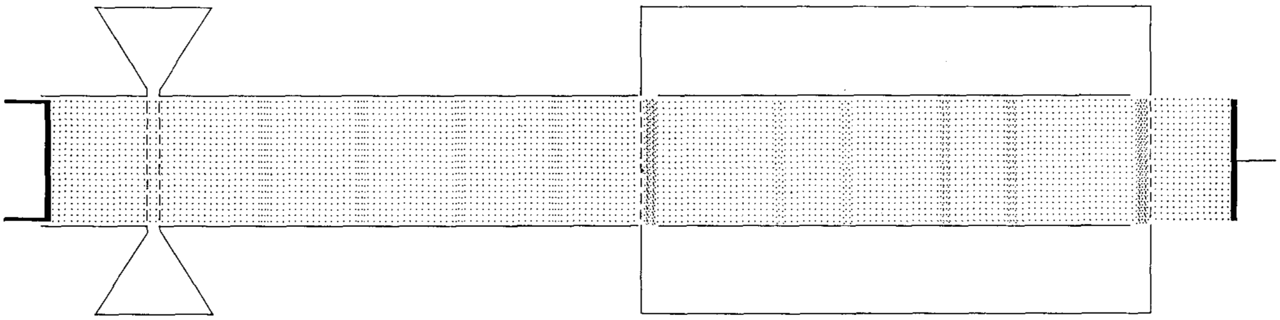


Figure 2 **Schematic diagram showing formation of bunches in drift-tube type frequency divider.**

drift angle. In this case one may imagine tight bunches of electrons entering the oscillator cavity at every half-cycle of the subharmonic oscillation. It is clear that such a bunch entering at time of zero subharmonic field will not couple to it at all. On the other hand, if the bunches enter at the times of maximum field amplitude (Fig. 2), alternate bunches will be accelerated and decelerated, leading to a current component in the beam at the subharmonic frequency during its second passage through the oscillator cavity gap. Furthermore, the subharmonic component is produced more efficiently in the prebunched beam than it would have been in a uniform beam carrying the same direct current. Thus one can see why a definite phase is preferred, and why the subharmonic can build up in the presence of prebunching even when the beam current is inadequate for it to build up in the absence of prebunching.

It might be noted here that even when the beam current exceeds the starting current, the prebunching, under certain circumstances, may still act to control the oscillator frequency and to fix its phase. For such operating conditions the device may be thought of as a locked oscillator rather than as a frequency divider. This distinction is probably not of any practical significance, however.[2,3]

2. Theoretical analyses

The operation of a klystron divider can be analyzed by methods typically applied in the study of klystrons. In the case of klystrons the effect of our harmonic electric field in the buncher gap on the motion of the beam electrons is used to compute the electronic current in the catcher gap. The ratio of the catcher current to the buncher voltage is then used to define an electronic admittance (or transadmittance, depending upon whether the buncher and catcher gaps do or do not coincide). A steady-state operating point, characterized by an amplitude and a frequency, is determined by requiring that the electronic admittance match an appropriately defined cavity admittance. In the case of the klystron divider, it is also necessary to take the effect of the prebuncher fields into account. This gives rise to an electronic admittance which depends upon the amplitude and phase of the prebuncher voltage. An operating point is again determined by requiring an impedance match. In the case of divider action, the frequency is determined by the prebuncher input; the operating point is characterized by an amplitude and phase.

Certain idealizations frequently made in the study of klystrons will be assumed in our discussion. These consist of the assumptions (1) that the velocity modulation is small compared to the mean velocity of the electrons, (2) that the drift spaces are field free, or, in the reflex case, have a constant retarding field, (3) that space charge forces are negligible. For the device envisaged, the condition (1) will be satisfied. While (2) may not be satisfied, the conclusions drawn will remain valid so long as certain of the drift angles are understood to be effective drift angles. Although the neglect of space-charge forces will probably not be justified for the device in question, it is hoped that the derived formulas will provide useful information.

• *a) Bunching theory*

We begin by computing the subharmonic component in the beam current at its second passage through the oscillator gap that is induced by a specified rf voltage at the input frequency at the prebunching gap and by an independently specified voltage at the subharmonic frequency at the oscillator gap.

Let t_1 be the time at which an electron passes through the prebunching gap, t_2 the time of its first passage through an oscillator gap, t_3 the time of its second passage through an oscillator gap. (The two passages may refer to the same or distinct oscillator gaps.) Let ω be the angular frequency of the subharmonic and 2ω that of the input. We take for the gap voltages $V_i \sin 2\omega t$ and $V_S \cos(\omega t+\beta)$. Then the times t_1, t_2, t_3, are related by

$$\omega t_2=\omega t_1+\theta_1-W_1 \sin 2\omega t_1$$
$$\omega t_3=\omega t_2+\theta_2-W_2 \sin 2\omega t_1-X \cos(\omega t_2+\beta). \qquad (1)$$

θ_1 and θ_2 are clearly the drift angles associated with the means of the time differences t_2-t_1 and t_3-t_2, respectively. W_1, W_2 and X are proportional to the respective voltage amplitudes at the prebunching and oscillator gaps. They may be written in the form

$$W_1=\frac{M_1V_i\theta_1'}{2V_0}; \quad W_2=\frac{M_1V_i\theta_2'}{2V_0}; \quad X=\frac{M_2V_S\theta_2'}{2V_0}.$$

Here V_0 is the dc voltage; M_1, M_2 beam coupling coefficients; θ_1', θ_2' effective drift angles. For the field-free drift tube case and the uniform repeller field case $\theta_1=\theta_1'$ and $\theta_2=|\theta_2'|$. The ratio W_2/W_1 is positive for the drift-tube case but negative for the reflex case.

The current component of frequency ω at the second oscillator gap passage is given by

$$i_S=\frac{M_2I_0}{\pi}\int_{-\pi}^{+\pi}e^{-j(\omega t_3)}\,d(\omega t_1)\,, \tag{2}$$

where I_0 is the beam current.

An electronic admittance may be defined by dividing i_S by the voltage at the oscillator gap. Thus

$$Y_e=\frac{i_S}{V_Se^{j\beta}}\,.$$

To evaluate the integral appearing in the expression (2) for i_S, one uses the relation (1) and the well-known Fourier series expansion of $\exp[jX\cos(\omega t+\phi)]$ (regarded as a function of ωt), obtaining

$$Y_e=\frac{jM_2^2|\theta_2'|}{2}\frac{I_0}{V_0}e^{-j\theta_2}$$

$$\sum_{p=0}^{\infty}\left\{(-1)^p\frac{2J_{2p+1}(X)}{X}[J_p(2pW_1-W_2)e^{j2p\gamma}\right.$$

$$\left.+J_{p+1}(2(p+1)W_1+W_2)e^{-j2(p+1)\gamma}]\right\}. \tag{3}$$

Just as is the case for the klystron oscillator, divider operation is characterized by various drift-angle modes. That is, operation is centered about values of θ_2 given by $\theta_2=2\pi(n+\frac{3}{4})$. For large values of θ_2', the variation of θ_2' and consequently W_2 for a given particular mode is unimportant and hence can be replaced by $\theta_{2,n}'$, the value of θ_2' at the center of the mode. Hence we write

$$Y_{e,n}=je^{-j\theta_2}G_{e,n}P_n(X,W,\gamma)\,,$$

with

$$G_{e,n}=\frac{I_0}{2V_0}M_2^2|\theta_{2,n}'|=\tfrac{1}{2}M_2^2|\theta_{2,n}'|G_0$$

$$W=W_1+|W_2|;\qquad W_1=\frac{\theta_{1,n}'W}{\theta_{1,n}'+|\theta_{2,n}'|};$$

$$W_2=\frac{\theta_{2,n}'W}{\theta_{1,n}'+|\theta_{2,n}'|}$$

$$P_n(X,W,\gamma)=\sum_{p=0}^{\infty}\left\{(-1)^p\frac{2J_{2p+1}(X)}{X}\times\right.$$

$$\left.[J_p(2pW_1-W_2)e^{j2p\gamma}+J_{p+1}(2(p+1)W_1+W_2)e^{-j2(p+1)\gamma}]\right\}.$$

The quantity $je^{-j\theta_2}G_{e,n}$ is precisely the small-signal admittance for a klystron oscillator. The n dependence of the quantity P_n occurs only through W_1 and W_2. In the special cases $W_1=0$ or $W_2=0$ it becomes independent of n.

• *b) Operation as a frequency divider*

Steady oscillations, of course, take place in a given mode when

$$Y_{e,n}=-Y_c(\omega)\,, \tag{4}$$

where $Y_c(\omega)$ is the admittance of the oscillator circuit. The quantities W, ω, n and θ_2 are all externally fixed operating parameters. The quantities which are determined by the Eq. (4) are, of course, the subharmonic level via X, and the phase via γ.

For a discussion of the conditions under which oscillations can begin, it is convenient to define a small-signal admittance

$$Y_{0,n}=je^{-j\theta_2}G_{e,n}P_n(0,W,\gamma)$$

$$P_n(0,W,\gamma)=J_0(W_2)+J_1(2W_1+W_2)e^{-2j\gamma}. \tag{5}$$ [4]

In Fig. 3, $P_n(0,W,\gamma)$ is plotted in the complex plane for a particular W_1, W_2 and a full cycle of γ. The curve is simply a circle of radius $J_1(2W_1+W_2)$ centered on the real axis a distance $J_0(W_2)$ from the origin. Also plotted on the same plane is the admittance ratio $j\frac{Y_c}{G_{e,n}}e^{j\theta_2}$ plotted for a range of values of θ_2 centered about $\theta_{2,n}$. Two such curves are plotted corresponding to different beam currents and hence to different values of $G_{e,n}$. These curves are simply arcs of circles centered at the origin. One of these arcs fails to intersect the $P_n(0,W,\gamma)$ circle and hence corresponds to a situation in which the beam current is too small to permit the subharmonic amplitude to build up. For the second arc, the two intersections with the $P_n(0,W,\gamma)$ curve determine the range of θ_2 about $\theta_{2,n}$ for which frequency division takes place. They also determine the range over which γ, or the phase of the subharmonic, varies as θ_2 is varied. Also plotted in Fig. 3 are curves of $P_n(X,W,\gamma)$ for constant X as γ is varied and for constant γ as X is varied. The two types of curves for convenience have been plotted in the upper and lower half-planes, respectively. The curves of constant X are, in fact, symmetric with respect to the real axis, while those of constant γ have a reflection in the upper half-plane corresponding to the opposite sign of γ. From the intersection of these curves with the admittance ratio arc, one can read off the relative power output (proportional to X^2) and phase as a function of θ_2. The results obtained are plotted in Fig. 4.

The minimum starting current in a given mode is, of course, determined by the maximum of $P_n(0,W,\gamma)$. For fixed W and $\theta_2'/(\theta_1'+|\theta_2'|)$ this maximum occurs either at $\gamma=0$ or $\gamma=\pi/2$, accordingly as $2W_1+W_2$ is positive or negative. In Fig. 5 this quantity, which is equal to $J_0(W_2)+J_1(|2W_1+W_2|)$ is plotted as a function of W for various values of $\theta_2'/(\theta_1'+|\theta_2'|)$, while in Fig. 6 its maximum in W is plotted as a function of $\theta_2'/(\theta_1'+|\theta_2'|)$.

• *c) Operation as a locked oscillator*

The starting condition for oscillation in a given mode, when the input voltage V_i is zero, is simply $G_{e,n}>G_c$, obtained essentially from (4) and (5) with $W=0$. It is

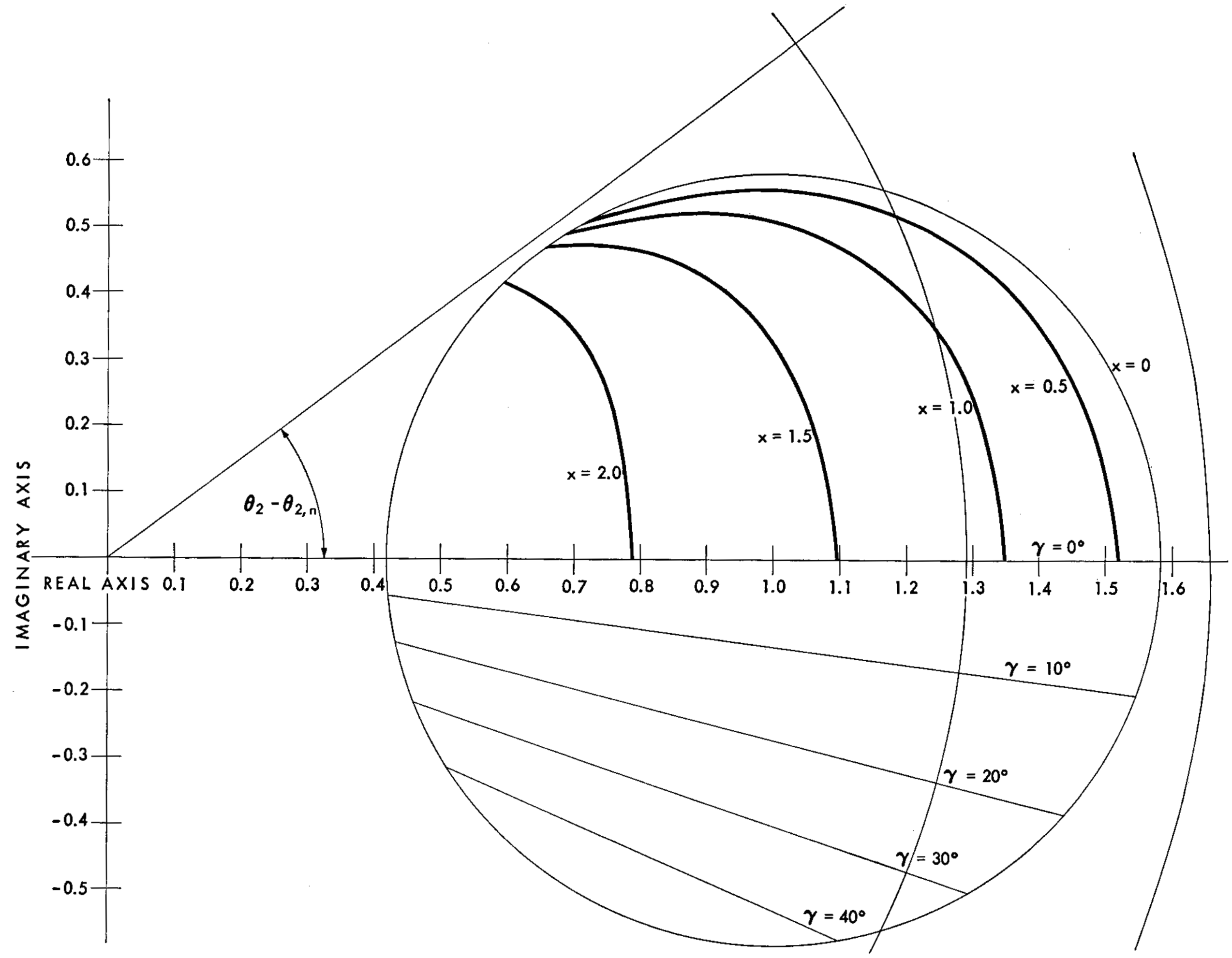
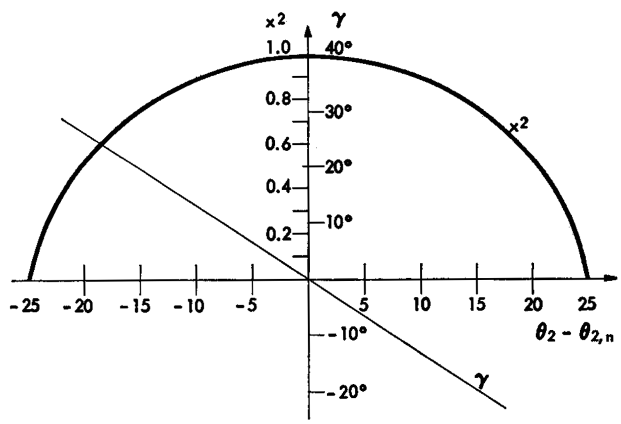


Figure 3 $P_n(X, \gamma)$ **for** γ **fixed and** X **varying, and for** X **fixed,** γ **varying for** $W_1=0.92$, $W_2=0$. **Also plotted:** $j(Y_c/G_{e,n})e^{j\theta_2}$ **for two different beam currents.**

apparent from Figs. 5 and 6 that a wide range of values for W and $\theta_2'/(\theta_1'+|\theta_2'|)$ exist for which operation as a frequency divider, rather than as a locked oscillator takes place. Thus, whenever the real part of $P(0, W, \gamma)$ exceeds unity, there exists a range of beam currents for which an amplitude can build up in the oscillator cavity only when V_i is sufficiently large. On the other hand, by increasing the input current one can certainly arrive at a point where ordinary klystron oscillator operation can take place. The question to be answered is whether or not the application of the input frequency will permit the oscillations to continue incoherently, or whether instead it will enforce a particular phase and frequency on these oscillations. It is clear, of course, that Eq. (4) can still be satisfied when $G_{e,n}$ exceeds the zero-W free oscillations threshold. Therefore, locked oscillations are certainly possible. Hence what we must first determine are the conditions under which free oscillations for non-zero W are also possible.

Figure 4 **Relative power output and phase as a function of** $\theta_2-\theta_{2,n}$.



The essential characteristic of free or incoherent oscillation is the fact that the frequency in the oscillator cavity is no longer precisely one-half that of the input frequency. A modification of the analysis in Part *a*), appropriate to this situation, follows:

Let ω' be the frequency of V_S, ω one-half the input frequency as before. If $\omega-\omega'$ is infinitesimal compared to ω, and if in addition one confines his attention to time intervals short compared to $2\pi/|\omega-\omega'|$, then the fact that $\omega-\omega'\neq 0$ has essentially no effect on the bunching process. On the other hand, the fact that ω is not equal to ω' makes itself felt over long time intervals via the phase γ, which is continually changing on account of this inequality. In terms of the discussion of Part *a*), we should write $\cos(\omega' t+\beta)$ as $\cos[\omega t+\beta'(t)]$ with $\beta(t)=\beta+(\omega'-\omega)t$ and carry through a derivation of the electronic admittance $Y_{e,n}$ ignoring the time dependence of β'(or $\gamma'=\beta'+\theta_1$). In the final expression for $Y_{e,n}$ γ' should, of course, replace γ so that the resultant electronic admittance is slowly varying, rather than constant. Hence one must expect that the incoherent oscillations are not constant in amplitude but are in fact amplitude modulated at the angular frequency $2(\omega-\omega')$. More important, we conclude that the average operating point is determined, at least approximately, by the condition

$$-Y_c(\omega')=\overline{Y}_{e,n}=je^{-j\theta_2}G_{e,n}\frac{2J_1(X)}{X}J_0(W_2)\,,$$

where $\overline{Y}_{e,n}$ means that the admittance has been averaged over γ. The average amplitude and the frequency ω' are then determined by this equation in a manner corresponding precisely to the ordinary klystron oscillator. Accordingly, the starting condition for these oscillations in a given mode is $G_c(\omega')<G_{e,n}J_0(W_2')$. Since $J_0(W_2)<1$ for all W_2 different from zero, it is apparent that a non-zero W_2 can act to suppress the free oscillations and hence that a region of locked oscillation exists.

It should be noted that the crude averaging we have used for estimating this suppression factor is valid only when the amplitude-modulation factor is small. If the circuit admittances at ω' and $2\omega-\omega'$ are comparable this will not be the case. Instead, operation at ω' will be accompanied by a comparable component at $2\omega-\omega'$ leading to typical two-frequency operation. For this situation the effective beam conductance can be nearly as large as in the case of locked oscillation or divider operation. The possible competition between divider operation and two-frequency operation should be studied, as it has an important bearing upon the question of phase stability. This question, however, will not be discussed here.

The origin of the suppression of free oscillations by prebunching can be traced to the phase aberration arising from the velocity modulation of the incoming beam. To elaborate, the electronic admittance for a beam of uniform density and velocity entering a klystron oscillator is given by

$$Y_e=jG_ee^{-j\theta}\frac{2J_1(X)}{X}\,.$$

If this incident beam contains a velocity and density distribution which is uncorrelated with the oscillator frequency, then it is appropriate to average Y_e over these distributions. The density occurs only in the factor G_e, and then linearly, so that the averaging simply yields the same result as using the mean density in G_e. On the other hand, the velocity modulation of the beam has an effect on the factor $e^{-j\theta}$ that is essentially the same as the typical reflex-klystron phase aberration arising from different electron transit times in different parts of the beam cross-section. Indeed, we should take

$$\int_{-\phi_m}^{\phi_m} e^{-j(\theta_0+\phi)}\rho(\phi)\,d\phi\,,$$

where $\rho(\phi)$ represents the relative density of electrons

Figure 5 **Reciprocal of relative starting current as a function of W for various values of $\eta=\theta'_{2,n}/(\theta_1+|\theta'_{2,n}|)$.**
(*Note: vertical scale begins at* 1.0)

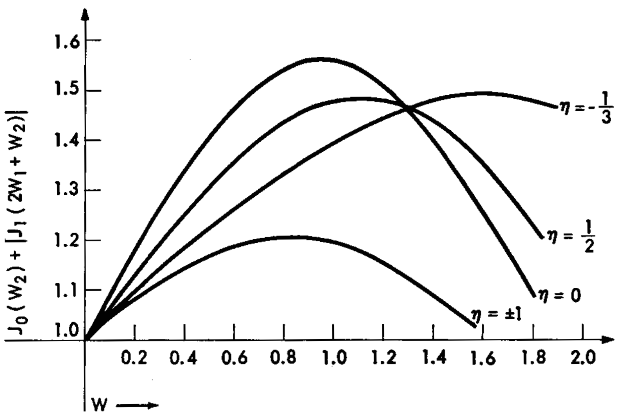
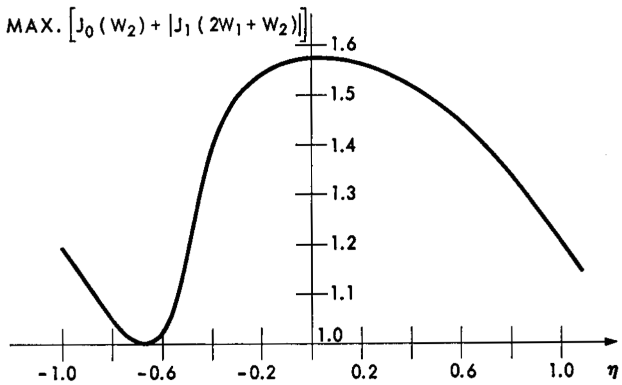


Figure 6 **Reciprocal of relative starting current as a function of η at its maximum in W.**
(*Note: vertical scale begins at* 1.0)



with the phase deviation ϕ from the mean phase ϕ_0, and ϕ_m is the maximum deviation. In the case of a sinusoidal modulation

$$\rho(\phi) = \frac{1}{\pi} \frac{1}{\sqrt{W_2^2 - \phi^2}},$$

where $W_2 = \phi_m$, and the above integral yields $e^{-j\theta_0} J_0(W_2)$. Hence the averaged admittance in the presence of prebunching is again found to be

$$Y_e = jG_e e^{-j\theta_0} J_0(W_2) \frac{2J_1(X)}{X}.$$

For the applications of this device to be discussed, there is no practical difference between the locked-oscillation region and frequency-dividing region, since the input voltage will be maintained steadily. Hence in the following sections this distinction will not be emphasized.

We have seen that as the beam current is increased from zero, the device passes from a condition of no-operation to one of frequency division and then to one of locked oscillation. The distinction between these latter two regions is one of terminology only, the designation "locked oscillation" being merely in recognition of the fact that free oscillations could take place in the absence of the input voltage. As the beam current is increased still further, however, the threshold for free oscillations in the presence of the input voltage is passed, and both types of operation become possible. Which type will take place in fact can be expected to depend upon the starting conditions. Since the applications of the device require rapid variations in the amplitude of the subharmonic, and since phase stability is essential, it would appear prudent to avoid operating conditions which could permit free oscillations. Limitations imposed by the requirement of suppression of two-frequency oscillation, however, may be more stringent than those implied by the above discussion.

3. Design considerations

The discussion of Section 2 applies to any klystron-type frequency divider. In this section some special problems associated with the proposed application will be discussed briefly.

In operation, the prebunching cavity will be provided with a constant input frequency 2ω. The oscillator cavity, however, will also be provided with an input of frequency ω, and amplitude A, zero, or $-A$. The phase of A will be so chosen that the two stable phases of the divider for zero subharmonic input just coincide with those of A and $-A$. There will be four distinct steady-state operations: one with the input A, another of equal amplitude and opposite phase with $-A$, and two of equal but smaller amplitudes and opposite in phase, associated with zero input. At any instant the input amplitude may change from one of its three values to another. In general, one wishes to minimize the time required to approach the new steady state, especially the switching time, i.e., the time required to change the sign of the amplitude in the oscillator cavity. The theory of the switching process will be discussed in Section 4. The essential features as they affect design, however, will be noted here.

The principal factors interfering with the rapid switching are the loaded Q of the oscillator cavity and the oscillator drift angle θ_2. The values of θ_2 associated with typical klystron-oscillator design would not be unsatisfactory for this application. Hence we will assume that the chief limiting feature is Q. The minimum switching time will be proportional to Q/ω. Hence the design should minimize this quantity. As an example we note that for $\omega/2\pi = 10$ kmc, a Q of 30 would be satisfactory, although a smaller value would be preferable. A loaded Q of 30 is considerably less than is in common use for low-level oscillators and the question arises as to whether such low Q's are possible.

As Q is reduced, the circuit admittance increases. This increase must be compensated for by an increase in the electronic admittance. Now

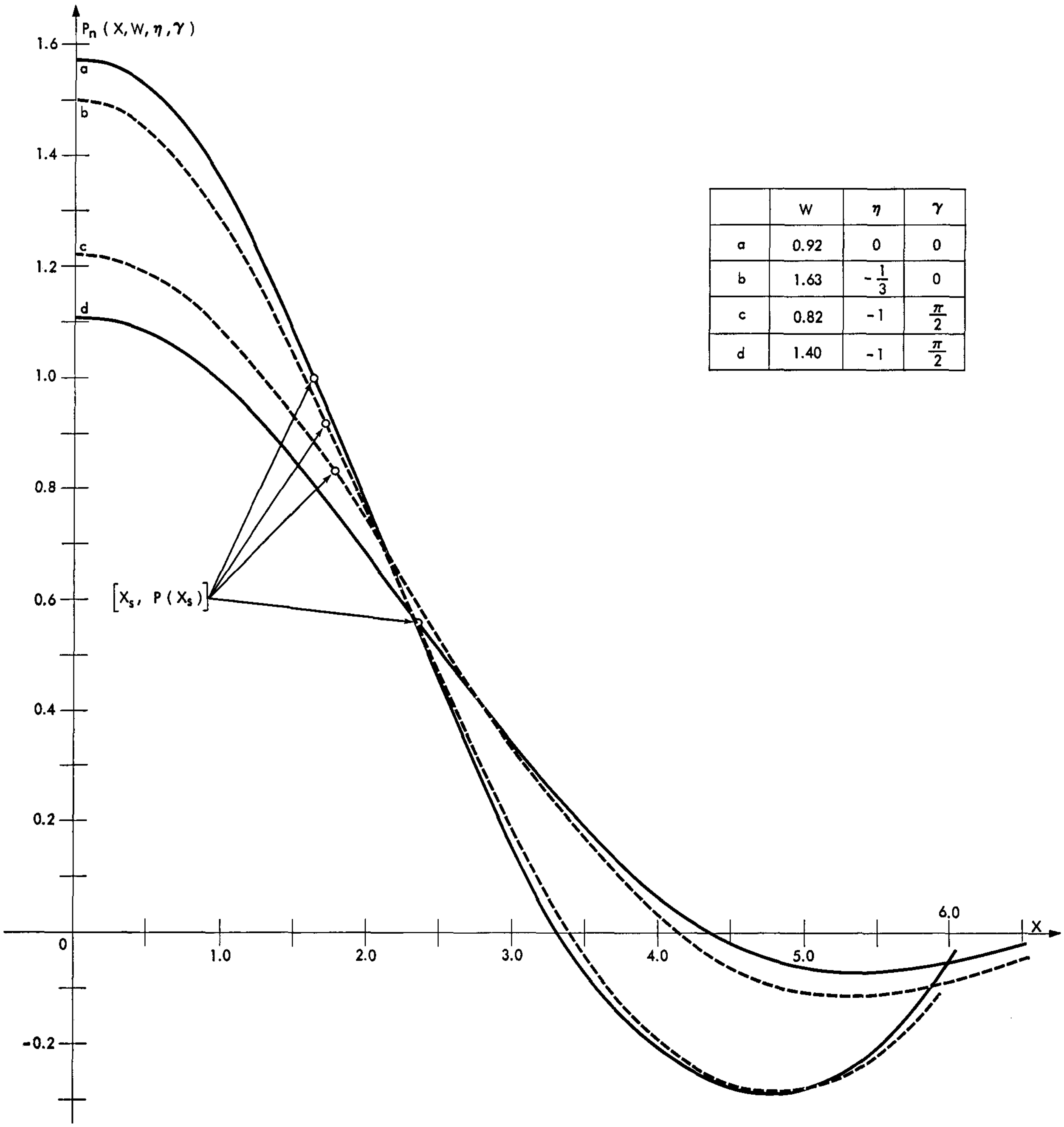
$$G_{e,n} = \tfrac{1}{2} M_2^2 |\theta'_{2,n}| G_0 .$$

Essentially no significant increase in M_2^2 is possible. In the case of the drift-tube klystron $\theta'_{2,n} = \theta_{2,n}$; hence very little increase in $\theta'_{2,n}$ is possible either, if this quantity is not itself to become a limiting factor in the switching time. Therefore, $G_{e,n}$ can be increased only by increasing G_0. The rate of increase of $G_{e,n}$ with G_0 is considerably decreased by the effects of space charge so that quite large values would be needed. This would in turn lead to a device with a rather high operating level. In the case of the reflex klystron, $|\theta'_{2,n}|$ is generally greater than $|\theta_{2,n}|$ owing to the fact that the electrostatic retarding field generally weakens as the reflector is approached. It is hoped that this effect can be considerably enhanced by means of careful design of the reflector field. One might hope in this way to achieve a satisfactory low-Q device operating in the milliwatt range. It might be noted in connection with this reference to a reflex design that it is probably important to avoid multiple electron transits, so that the usual precautions to avoid them should be incorporated into the design of the reflecting field.

The switching time may also be affected by the magnitude of the subharmonic input A. Indeed, for switching to occur at all, this amplitude must exceed some critical value (see Section 4). Furthermore, for an amplitude A which exceeds the critical value by an arbitrarily small amount, the switching time can be arbitrarily long. On the other hand, an unnecessarily large value of A may lead to an inconveniently large difference between the operating amplitude obtained with input A versus that obtained with zero input. Both of these amplitudes are determined by the behavior of the electronic admittance as a function of X, and hence by the behavior of $P_n(X, W, \gamma)$. In operation, $\theta'_{2,n}$ would generally be chosen so that $\gamma = 0$ or $\pi/2$ accordingly as $2W_1 + W_2$ is positive or negative. Also, W would be chosen large enough to maximize the small signal admittance, or perhaps in certain cases, larger. In Fig. 7, $P_n(X, W, \genfrac{}{}{0pt}{}{0}{\pi/2})$ is plotted as a function of X for values of $\theta'_{2,n}/(\theta'_{1,n} + |\theta'_{2,n}|)$ which may be of practical significance, and for the indicated values of W, chosen either equal to or larger than the value necessary to maximize the small-signal admittance. On each curve, a point $[X_S, P(X_S)]$ is marked, corresponding to the relation $J_0(W_2)=P_n(X_S, W, \genfrac{}{}{0pt}{}{0}{\frac{\pi}{2}})$. The significance of $J_0(W_2)$ in connection with the starting condition for free oscillations has been noted in the previous section. The condition that parameters be chosen such that free oscillations be impossible implies that X_0, the value of X in the absence of a subharmonic input, must be less than X_S. On the other hand, to take full advantage of the low Q of the oscillating circuit, and provide rapid switching, it is necessary that X_A, the value of X in the presence of the input amplitude A, be such that $P_n(X_A, W, \genfrac{}{}{0pt}{}{0}{\frac{\pi}{2}})\leq 0$. Under these conditions the electron beam is no longer driving the subharmonic but is instead acting as a load. Hence the orders of magnitude of X_0/X_A can be read off these curves for the various cases.

We conclude with a remark about the choice of $\theta'_{2,n}/(\theta_{1,n}+|\theta'_{2,n}|)$. There is no doubt that, electrically, the optimum choice for this parameter is zero (or near

Figure 7 $P_n(X, W, \eta, \genfrac{}{}{0pt}{}{0}{\frac{\pi}{2}})$ **as function of** X **for given** W, η, γ **values. For** $X_S : J_0(W_2)=P_n(X_S, W, \eta, \genfrac{}{}{0pt}{}{0}{\frac{\pi}{2}})$.



	W	η	γ
a	0.92	0	0
b	1.63	$-\frac{1}{3}$	0
c	0.82	-1	$\frac{\pi}{2}$
d	1.40	-1	$\frac{\pi}{2}$

zero). To achieve such a value, however, requires a very long prebunching drift space. Particularly if this choice leads to a loss of beam current, some compromise should be made. The choice $\eta \cong -1$ for the reflex case is, practically speaking, a very convenient one, and the added convenience might compensate for the somewhat poorer intrinsic characteristics.

4. Transient response of the frequency divider

In Section 3 of this paper the main ideas and results in connection with the switching time have been already discussed. We shall present a more detailed analysis of this subject here.

The discussion will be based upon energy-balance considerations, assuming steady-state properties for the passive circuit elements. While transient effects in these elements are ignored chiefly in the interest of simplicity, it is expected that they would be unimportant in the proposed application.

During the transient state the voltage-amplitude V_S of the divider cavity, and hence the energy, is a function of time. It is convenient to take the bunching parameter X, which is proportional to V_S, as a measure of the amplitude.

The stored energy in the cavity at any time is c_1X^2, where c_1 is a proportionality factor. Therefore the rate at which the energy-content of the cavity changes is $2c_1X\dot{X}$, where the differentiation refers to time. As a natural dimensionless time variable for this problem we take $\tau = \omega t/2Q$, because the switching time is proportional to Q/ω. (Q is the loaded Q of the divider cavity.) The change of the energy content in the cavity occurs for the following reasons:

1) The cavity is not perfectly conducting and may be loaded. The resultant resistive loss rate of the cavity is given by c_2X^2, where c_2 is a positive constant. We may say that energy is produced at the rate of $-c_2X^2$.

2) The external source delivers energy. We take it to be of the constant-current type, so that the rate at which it delivers energy is given by c_3X, where c_3 is a constant.

3) The bunched beam also delivers energy. The current of the beam is proportional to $X^*P(X^*)$ where X^* stands for $X(\tau-\theta_2/2Q)$. Therefore the rate at which the bunched beam delivers energy is given by $c_4XX^*P(X^*)$, where c_4 is a proportionality factor. Thus we obtain the equation

$$2c_1X\dot{X} = -c_2X^2+c_3X+c_4XX^*P(X^*) .$$

The trivial and unstable solution $X=0$ should be eliminated by dividing by X. The constants can conveniently be expressed in terms of various parameters. First of all, in order that the Q appearing in the definition of τ correspond to the loaded Q it is necessary that $2c_1=c_2$. Then $c_3/2c_1$ will be determined by defining as a unit source amplitude that amplitude which yields X_1 as the equilibrium amplitude, where X_1 is the first positive root of $P(X)=0$. Thus we write $c_3/2c_1=\alpha X_1$ where α is the source amplitude in the unit defined above. Thus $c_4/2c_1$ can be expressed in terms of the steady state amplitude, X_0 attained in the absence of an external source. With these definitions we obtain

$$\dot{X} = -X+\alpha X_1+\frac{X^*P(X^*)}{P(X_0)} . \tag{6}$$

We first apply this equation to the steady state problem, in which case we have $\dot{X}=0$; $X^*=X$ and

$$X-\alpha X_1 = \frac{XP(X)}{P(X_0)} .$$

This relationship is illustrated in Fig. 8, where the left and right sides are plotted as functions of X for various values of α. The intersections correspond to solutions of this equation. For $|\alpha|>0.07=\alpha_{\text{crit}}$ there is one solution. For $|\alpha|<0.07$ there are three intersections, corresponding to two stable and one unstable (the central intersection) operating points. In particular $X=\pm X_0$ are the possible operating points for $\alpha=0$. In order that the device operate as described in the text it is necessary that the applied signal exceeds the critical value ($|\alpha|>\alpha_{\text{crit}}$).

The transient behavior is obtained by integrating Eq. (6) numerically. For a numerical solution neither the fact that $P(X)$ must be evaluated numerically nor the retardation implicit in X offers any complication. A minimum switching time, τ_S, can be defined with reference to the following problem.

The divider is assumed to be operating in the steady state corresponding to an input $-\alpha<-0.07$. The input $+\alpha$ is then applied for a time τ_S, after which no input is applied. There exists a time τ_S such that for $\tau<\tau_S$, the steady-state amplitude finally is $-X_0$, while for $\tau>\tau_S$ it is $+X_0$. Clearly τ_S depends principally on α and in particular becomes long as α approaches α_{crit}.

An example of the transient behavior relevant to the determination of τ_S is illustrated in Fig. 9. In this case the parameters have been chosen with the view of obtaining a small τ_S. We have taken $X_0=X_S$ (see Fig. 7c and related text), the retardation parameter $\theta_2/2Q=0.55$, and $\alpha=1.25$. The three curves correspond to different values of τ and we note that $\tau_a<\tau_S$ but both τ_b, τ_c exceed τ_S. Then τ_S is approximately 0.8. This small value is obtained at the cost of a switching power approximately eight times as large as the power output of the divider operating without input. As the discussion in Section 3 indicates, reduction of the switching power below that which yields an optimum switching time leads to an increase in switching time. There would, for example, be some practical advantage in using a switching power equal to the output of the divider operating without input ($\alpha=0.2$). For this input, however, one finds $\tau_S \cong 6$, so that the increase in switching time is quite large.

Appendix: Frequency division by factors greater than two

A bunching theory appropriate to the case of frequency division by an arbitrary integer m can be developed in a manner identical to that given in Section 2a) for the case $m=2$. Starting from the transit time relations

$$\omega t_2=\omega t_1+\theta_1-W_1 \sin m\omega t_1$$

$$\omega t_3=\omega t_2+\theta_2-W_2 \sin m\omega t_1-X \cos(\omega t_2+\beta)\,,$$

one finds

$$Y_e=\frac{2jG_e}{X}\, e^{-j\theta_2}\left\{\sum_{l=0}^{\infty} j^{l(m+2)} J_l(W_2-mlW_1)\, J_{1+ml}(X)\, e^{jlm\gamma}\right.$$

$$\left.-\sum_{l=1}^{\infty} j^{lm} J_l(W_2+mlW_1)\, J_{ml-1}(X)\, e^{-jlm\gamma}\right\}.$$

The most important feature of this expression becomes apparent when one considers the small-signal admittance for $m>2$. In this case

$$Y_{e,0}=jG_e e^{-j\theta_2} J_0(W_2) \qquad m>2\,.$$

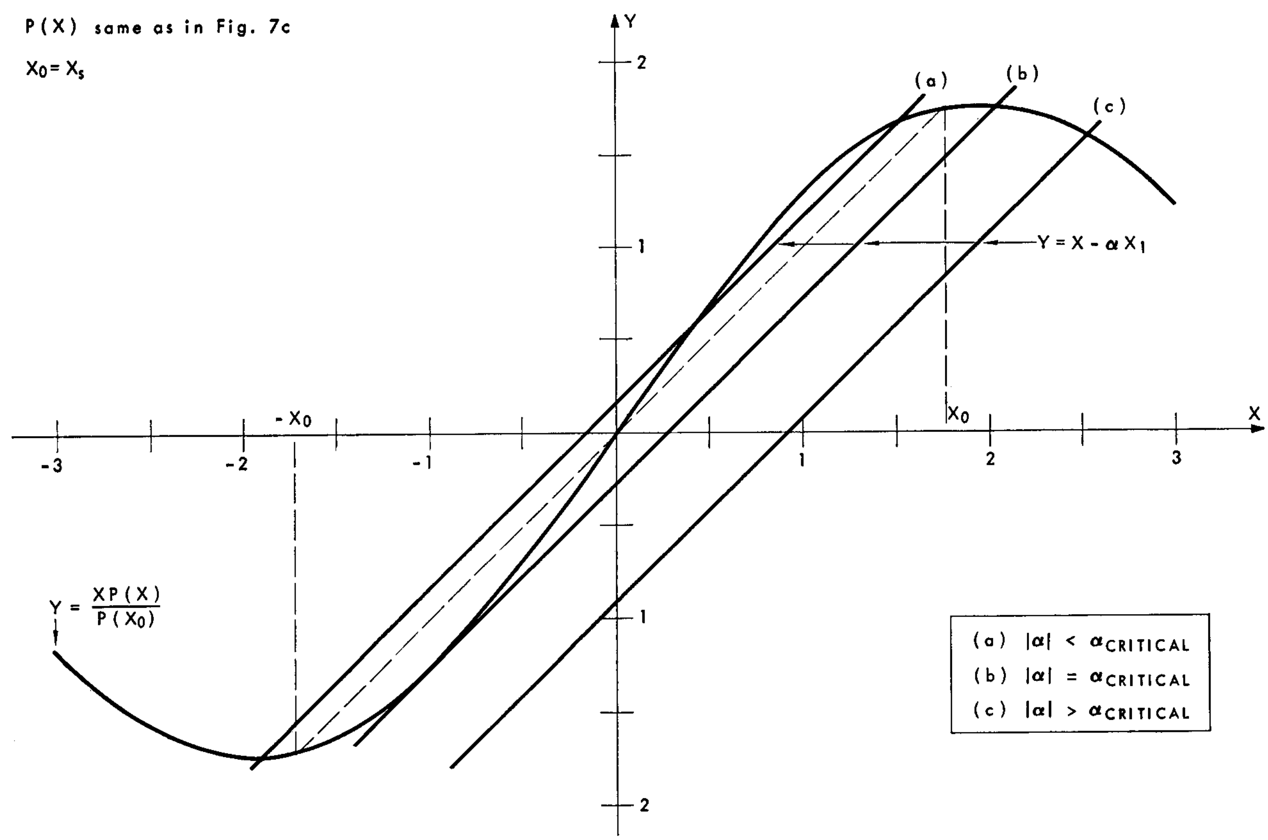
This expression is independent of m and of the relative phase γ and is identical with the averaged small-signal admittance discussed in Section 2c). From this result one can conclude that for $m>2$ there is no region of frequency division or locked oscillation in the sense discussed in Section 2. That is, with the beam current sufficiently large to permit the build-up of the subharmonic amplitude from an infinitesimal level, free oscillations are also possible. There may, of course, be conditions under which the device will prefer to lock rather than oscillate freely, but more detailed theoretical consideration would be required to see what these conditions might be.

Of course, there is the possibility that $|Y_e|$ may exceed its small-signal value for nonzero values of X. This will certainly be the case if W_2 is so chosen as to make $J_0(W_2)$ very small, or if $m=3$ and $J_1(3W_1+W_2)$ does not vanish. In this case beam currents which are not sufficient to permit the build-up of the subharmonic from zero amplitude, or to sustain free oscillation, may well be sufficient to sustain a subharmonic amplitude which has been built up to an appropriate level by an external source.

References

1. Present paper first appeared as an IBM Watson Laboratory Report dated Sept. 15, 1957.
2. Operation of a reflex klystron as a frequency divider has been described by E. N. Bazarov, M. E. Zhabativski, "Frequency Division with Reflex Klystron" *Radiotechnika & Electronika* No. 5, p. 680 (1956); T. J. Bridges, "A parametric electron beam amplifier," *Proc. IRE* **46,** No. 2, 494 (February 1958).
3. In addition, a device similar to that discussed here has been described by A. Ashkin, T. J. Bridges, W. H. Louisell, and C. F. Quate, "Parametric Electron-Beam Amplifiers," *1958 IRE WESCON Convention Records,* Part 3, p. 13.

Figure 8 **Steady-state amplitudes of the divider for various voltage inputs.**



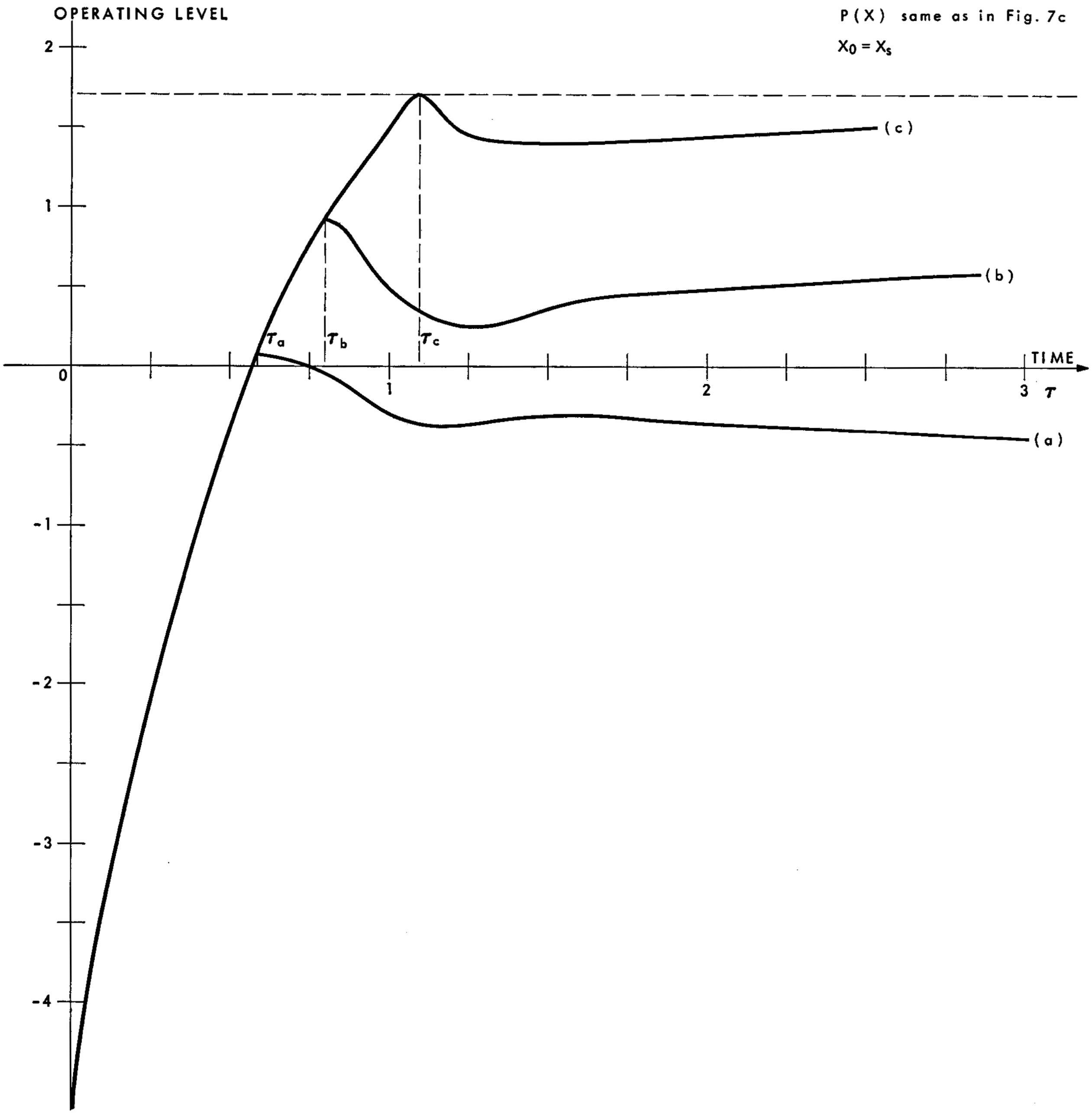


Figure 9 **Operating level of the divider as a function of time. The input voltage is turned off at τ_a, τ_b and τ_c, respectively.**

They describe their device as a variable-parameter amplifier, and choose the klystron drift angle in such a way that the varied parameter (the electronic admittance) is susceptive. From the variable-parameter point of view our device is a variable negative-conductance oscillator. For a fast-switching application, the more efficient use of the beam electrons associated with the variable negative-conductance mode of operation is advantageous.

4. The "variable susceptance" operation of the Ashkin, et al device referred to in Footnote 3 is characterized by a θ_2 chosen so as to make the quantity $je^{-j\theta_2}$ appearing in Eq. (5) imaginary, while in our discussion to follow, we choose θ_2 so as to make this quantity negative and real (corresponding to "variable negative conductance"). The maximum effective negative conductance given by our theory for the variable-susceptance case is $G_{e,n}J_1(2W_1+W_2)$, while for the variable negative conductance case it is $G_{e,n}[J_0(W_2)+J_1(2W_1+W_2)]$. As will be discussed in Section 3, minimization of switching time, which has been our dominant design consideration, requires maximization of the electronic negative conductance. As discussed by Ashkin et al, variable-susceptance operation has advantages for the low-noise amplification application which they had principally in mind.

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