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A Gas Film Lubrication Study Part II Numerical Solution of the Reynolds Equation For Finite Slider Bearings

Abstract: This paper presents a finite-difference technique for obtaining approximate numerical solutions to the Reynolds partial differential equation of gas film lubrication theory. A digital computer program is described, and discretization errors and stability of the difference equations are discussed.

Introduction

The work reported here was undertaken in support of two distinct activities which are described in Part I and Part III of this series.

The principal analytical tool for the investigation of film lubrication is a partial differential equation known as the Reynolds equation. Part I deals with a derivation of this equation. Since in the most general case the Reynolds equation has not been solved in closed form, recourse must be made to approximate numerical solutions. The present paper describes a technique by which numerical solutions were obtained with the aid of a digital computer.

The large amount of computation which has been performed thus far has served a twofold purpose. On the one hand it reveals the general behavior, both static and dynamic, of gas-lubricated bearings under a wide range of conditions. These results are presented in Part I. On the other hand it serves as a guide to the optimum design of specific bearings, supplementing experimental work and in some cases even supplanting it. This is dealt with in Part III.

Derivation of the finite difference approximation to the Reynolds equation

The Reynolds equation for the pressure $p(x, y)$ at a point (x, y) in a compressible lubricating film is

$$[(h^3p^{1/n}p_x)/\mu]_x+[(h^3p^{1/n}p_y)/\mu]_y=6U(hp^{1/n})_x\,, \quad (1)$$

where μ is the viscosity, U the speed of the moving surface, $h=h(x, y)$ the film thickness, and n the exponent in the polytropic gas law

$$p\rho^{-n}=\text{constant},\ 1\leq n\leq k\,.$$

The subscripts x and y denote differentiation in the x and y directions. (For derivations and further discussion of the equation, the reader is referred to Part I of this series.)

Assuming the viscosity to be independent of the coordinates x and y, and carrying out the indicated differentiations, we may write Eq. (1) in the form

$$p(p_{xx}+p_{yy})=\left(\frac{6\mu U}{nh^2}-\frac{3ph_x}{h}\right)p_x-\frac{3ph_y}{h}\,p_y-\frac{1}{n}\,(p_x{}^2+p_y{}^2)+6\mu U\,\frac{ph_x}{h^3}\,. \quad (2)$$

We seek numerical solutions to Eq. (2) over a rectangular region of the x, y plane. It will be convenient to take as the boundary of the rectangle the lines $x=0$, $x=B$, $y=L/2$, and $y=-L/2$. We shall allow the film thickness function $h=h(x, y)$ to be arbitrary except for the symmetry condition $h(x, y)=h(x, -y)$. Since this symmetry implies a symmetry in the pressure function $p(x, y)=p(x, -y)$, solutions will be obtained only for the upper half of the rectangle.

The boundary value problem, then, is to solve Eq. (2) in the half rectangle bounded by the lines $x=0$, $x=B$, $y=0$, and $y=L/2$, with boundary conditions

$$p(0, y) = p_a$$
$$p(B, y) = p_a$$
$$p(x, L/2) = p_a$$
$$p_y(x, 0) = 0\,,$$

where p_a denotes ambient pressure.

Consider a net formed by lines parallel to the y axis spaced a distance $\Delta x = B/M$ apart and by lines parallel to the x axis spaced at $\Delta y = L/2N$, where M and N are positive integers. Let (x_k, y_m) be an arbitrary point of the mesh and let $p_{k,m}$ be defined by

$$p_{k,m} = p(k\Delta x, m\Delta y) \quad (k=0,1,\ldots,M;\ m=0,1,\ldots,N).$$

If the pressure function $p(x, y)$ be expanded in a Taylor series about the point (x_k, y_m), we obtain in the usual way the following formulas for the derivatives of p at (x_k, y_m):

$$p_x(x_k, y_m) = \frac{p_{k+1,m} - p_{k-1,m}}{2\Delta x} - \frac{(\Delta x)^2}{6}\, p_{xxx}(\xi, y_m)$$

$$p_y(x_k, y_m) = \frac{p_{k,m+1} - p_{k,m-1}}{2\Delta y} - \frac{(\Delta y)^2}{6}\, p_{yyy}(x_k, \eta)$$

$$p_{xx}(x_k, y_m) = \frac{p_{k+1,m} - 2p_{k,m} + p_{k-1,m}}{(\Delta x)^2} - \frac{(\Delta x)^2}{12}\, p_{xxxx}(\xi', y_m)$$

$$p_{yy}(x_k, y_m) = \frac{p_{k,m+1} - 2p_{k,m} + p_{k,m-1}}{(\Delta y)^2} - \frac{(\Delta y)^2}{12}\, p_{yyyy}(x_k, \eta')\,,$$

where $x_{k-1} < \xi,\ \xi' < x_{k+1}$ and $y_{m-1} < \eta,\ \eta' < y_{m+1}$.

If we substitute these expressions into Eq. (2) and collect like powers of $p_{k,m}$ we obtain, after a straightforward but tedious calculation,

$$p_{k,m}{}^2 - 2G_{k,m}p_{k,m} + H_{k,m} = 0\,,$$

where

$$G_{k,m} = G_{k,m}(p_{k+1,m};\ p_{k-1,m};\ p_{k,m+1};\ p_{k,m-1})$$

and

$$H_{k,m} = H_{k,m}(p_{k+1,m};\ p_{k-1,m};\ p_{k,m+1};\ p_{k,m-1})$$

are independent of $p_{k,m}$. Specifically,

$$\begin{aligned} 2G_{k,m} = {} & \frac{p_{k+1,m} + p_{k-1,m}}{\Gamma(\Delta x)^2} + \frac{p_{k,m+1} + p_{k,m-1}}{\Gamma(\Delta y)^2} \\ & + \frac{3h_x(p_{k+1,m} - p_{k-1,m})}{2\Gamma(\Delta x)h} + \frac{3h_y(p_{k,m+1} - p_{k,m-1})}{2\Gamma(\Delta y)h} \\ & - \frac{6\mu U h_x}{\Gamma h^3} - \frac{(\Delta x)^2}{\Gamma}\left(s + \frac{3h_x}{h}\,v\right) \\ & - \frac{(\Delta y)^2}{\Gamma}\left(t + \frac{3h_y}{h}\,w\right), \end{aligned} \tag{3}$$

$$\begin{aligned} H_{k,m} = {} & \frac{3\mu U}{\Gamma h^2(\Delta x)n}\,(p_{k+1,m} - p_{k-1,m}) \\ & - \frac{1}{4\Gamma(\Delta x)^2 n}\,(p_{k+1,m} - p_{k-1,m})^2 \\ & - \frac{1}{4\Gamma(\Delta y)^2 n}\,(p_{k,m+1} - p_{k,m-1})^2 - \frac{6\mu U}{\Gamma h^2 n}\,v(\Delta x)^2 \\ & + \frac{p_{k+1,m} - p_{k-1,m}}{\Gamma n}\,v(\Delta x) + \frac{p_{k,m+1} - p_{k,m-1}}{\Gamma n}\,w(\Delta y) \\ & - \frac{v^2(\Delta x)^4}{\Gamma n} - \frac{w^2(\Delta y)^4}{\Gamma n}\,, \end{aligned} \tag{4}$$

where

$$\Gamma = \frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}\,,$$

$$s = (1/12)\,p_{xxxx}(\xi', y_m)\,, \qquad t = (1/12)\,p_{yyyy}(x_k, \eta')\,,$$
$$v = (1/6)\,p_{xxx}(\xi, y)\,, \qquad w = (1/6)\,p_{yyy}(x_k, \eta)\,.$$

The finite difference approximation to Eq. (2) is then obtained by omitting from Eqs. (3) and (4) those terms involving the higher derivatives, s, t, v, and w. Thus we obtain an approximation $\bar{p}_{k,m}$ to $p_{k,m}$ which satisfies

$$\bar{p}_{k,m}{}^2 - 2\overline{G}_{k,m}\bar{p}_{k,m} + \overline{H}_{k,m} = 0\,, \tag{5}$$

where $\overline{G}_{k,m}$ and $\overline{H}_{k,m}$ are obtained from $G_{k,m}$ and $H_{k,m}$ by setting $s = t = v = w = 0$. Solving Eq. (5) for $\bar{p}_{k,m}$, we obtain

$$\bar{p}_{k,m} = \overline{G}_{k,m} + \sqrt{\overline{G}_{k,m}{}^2 - \overline{H}_{k,m}}\,. \tag{6}$$

On the other hand, by retaining terms in s, t, v, and w, we may readily verify that the local truncation error $p_{k,m} - \bar{p}_{k,m}$ is $O[(\Delta x)^2] + O[(\Delta y)^2]$.

It should be remarked that the other root of the quadratic equation (5) is extraneous, as may be shown by the following continuity argument. If we set $u = 1/n$ in Eq. (4), then

$$\overline{H}_{k,m} = u\overline{H}'_{k,m}\,,$$

where $\overline{H}'_{k,m}$ is independent of u. Setting $u=0$ (incompressible flow) we have $\overline{H}_{k,m} \equiv 0$, and consequently, $\bar{p}_{k,m} = 2\overline{G}_{k,m}$. This relation together with the requirement that $\overline{H}_{k,m}$, and therefore $\bar{p}_{k,m}$, be a continuous function of u establishes Eq. (6).

Numerical solution of the finite difference equation

Numerical solutions of Eq. (5) have been obtained by the "Extrapolated Liebmann" method,[1] a single-step iterative process in which the net is traversed in a fixed sequence with the old values of $p_{k,m}$ being replaced by the new as soon as they are obtained. If the mesh is being traversed for the $(j+1)^{\text{th}}$ time, then the new value is the extrapolated value

$$p^{(j+1)} = \gamma q^{(j+1)} + (1-\gamma)p^{(j)}\,, \tag{7}$$

where $q^{(j+1)}$ denotes the right-hand member of Eq. (6) and where γ is a constant greater than unity. The iterations are continued until the "convergence indicator,"

$$\sum_{k,m}\left|p_{k,m}^{(j+1)}-p_{k,m}^{(j)}\right| \Big/ \sum_{k,m}\left|p_{k,m}^{(j+1)}\right|, \tag{8}$$

is sufficiently small.

The boundary condition that $p_y=0$ for $y=0$ is approximated by setting $p_{k,-1}=p_{k,1}$ when calculating $p_{k,0}$. The net is traversed in the sequence $(1,0)$, $(1,1)$, $(1,2), \ldots,$ $(1, N-1)$, $(2,0)$, $(2,1)$, etc. Experiments with several more complicated sequences showed this simple sequence to produce the most rapid convergence.

The total load, W, on the slider bearing is

$$W=\int_{-L/2}^{L/2}\int_0^B [p(x,y)-p_a]dxdy$$

and the x coordinate of the center of pressure is

$$x_c=\frac{1}{W}\int_{-L/2}^{L/2}\int_0^B x[p(x,y)-p_a]dxdy. \tag{9}$$

Because of symmetry the y coordinate is, of course, zero.

These integrals are computed approximately by applying Simpson's rule to the $p_{k,m}$. For brevity, let

$$P_{k,m}=p_{k,m}-p_a.$$

Then, with $y=m(\Delta y)$, we have

$$\int_0^B [p(x,y)-p_a]dx=\frac{\Delta x}{3}(P_{0,m}+4P_{1,m}+2P_{2,m}+\ldots\ldots+4P_{M-1,m}+P_{M,m})-\frac{(\Delta x)^5 M}{90}E, \tag{10}$$

where $|E|<\max\limits_{0<x<B} p_{xxxx}$.

The fourth derivative of p is, of course, unknown; but we may estimate it by considering the special case of incompressible flow under a slider bearing of infinite length. Setting $h(x)=\alpha(A-x)$, the Reynolds equation yields the exact solution

$$p(x)=\frac{6\mu Ux(x-B)}{\alpha^2(B-2A)(A-x)^2}.$$

The fourth derivative of $p(x)$ is readily calculated, and moreover, it is easily verified that its maximum value occurs at $x=0$. We thus find that

$$|E|<\frac{144\mu U|3A-4B|}{A^5\alpha^2(2A-B)}.$$

For a typical case, if we choose $B=0.578$ inches, $\alpha=0.0003$ radians, $\mu=2.62\times10^{-9}$ lb sec/in^2, $U=1500$ in/sec, $M=12$, and $a=0.911$ in, we find that

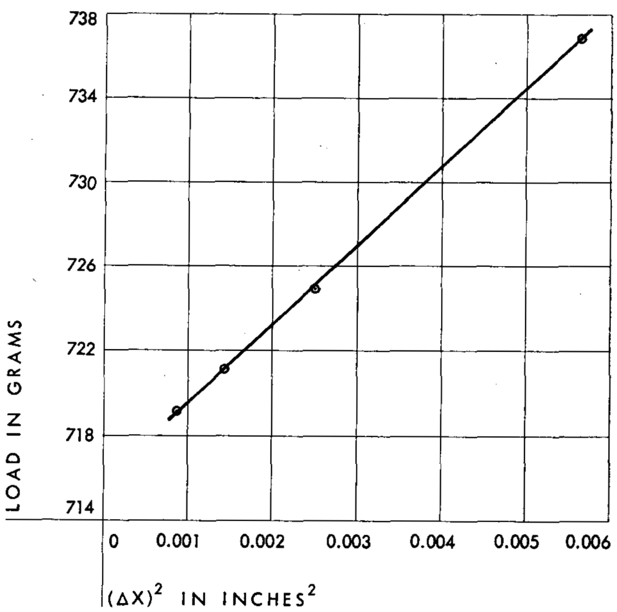
$$\frac{(\Delta x)^5 M}{90}E<0.0004.$$

For this set of parameters the computer solution gives approximately 2.0 atmospheres as an average value of $p_{k,m}$, and hence, the integral on the left-hand side of Eq. (10) is approximately 1.2. The error is thus seen to be less than 0.0004/1.2, or about 0.03 per cent. For the majority of our computations this error is considerably smaller.

A similar analysis carried out for the integral (9) yields comparable results. It should be emphasized that these estimates of error apply to the integration process (and in fact only to the integration in the x direction) and have nothing to do with the discretization error which arises from replacing the continuous problem (the Reynolds equation) by the discrete problem (the finite difference equation) or with the error which arises in solving numerically the discrete problem.

In nonlinear problems of the type we are concerned with here, precise analysis of these errors lies beyond the presently known techniques of numerical analysis. However, some insight can be gained by studying experimentally the behavior of the load, W, and the position of the center of pressure x_c as Δx is decreased while holding the ratio $\Delta x/\Delta y$ constant. This was carried out on the computer using 28, 66, 120, and 190 mesh points, in each case the iterations being continued until there was practically no change in the values of $p_{k,m}$. The results are shown in Figs. 1 and 2, where W and x_c are plotted against $(\Delta x)^2$. The straight lines are obtained by a least squares

Figure 1 **Load, W, plotted against $(\Delta x)^2$ for a cylindrical surface bearing.**
(Breadth = 0.6 in., length = 0.55 in., crown height = 250 μin.; velocity = 2500 ips; min. film thickness = 250 μin.; viscosity = 2.76 × 10^{-9} lb. sec./in.^{2}; angle = 1.5 × 10^{-3} radians; $\Delta x/\Delta y$ = 1.091.)



fit. Figure 1 suggests that discretization error affects the load, W, to the order of $(\Delta x)^2$, while the error in x is somewhere between $O(\Delta x)$ and $O[(\Delta x)^2]$.

For some mesh ratios $\Delta x/\Delta y$ or for excessively large values of γ, the extrapolation constant in Eq. (7), the iterative scheme becomes unstable. Both weak instability (small but bounded oscillations in successive values of $p^{(j)}$) and strong instability (oscillations of rapidly increasing magnitude) occur. Instability can be dealt with in either of two of the following ways or by a combination of both.

1. The extrapolation constant γ may be reduced. Normally, γ in the range 1.4 to 2.0 produces the most rapid convergence. Severe cases of instability may sometimes be cured by taking γ to be less than unity.
2. The mesh ratio $\Delta x/\Delta y$ may be increased.

The computer program

The procedure described in the preceding sections was carried out on the IBM 650 Magnetic Drum Data Processing Machine. This section is devoted to a general description of the computer program. Detailed flow diagrams and program listings are available from the author.

Much of the computational work was carried out in support of the experimental investigations which are reported in Part III. For this purpose it was sufficient to assume for the film thickness $h(x,y)$ a quadratic expression

$$h(x, y) = c_0 + c_1 x + c_2 x^2 + c_3 y^2 .$$

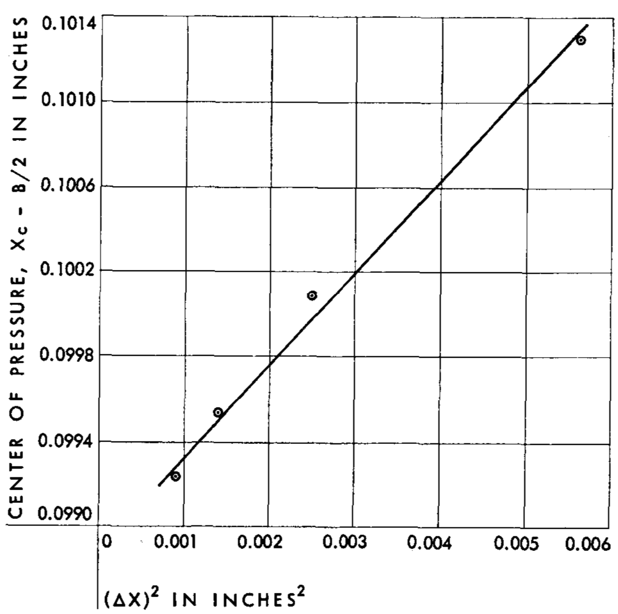
With suitable values of the coefficients this function accurately represents plane, cylindrical, and spherical slider bearing surfaces at arbitrary inclination angles. The computer program is designed to be as fully automatic as possible. All the necessary input information is entered on two cards. This information includes the dimensions of the slider, its surface shape, pivot location, speed, minimum clearance desired, and an estimate of the inclination angle. The true angle, which is usually unknown, is of course the equilibrium angle for the pivoted slider, i.e., the angle for which the center pressure lies on the pivot axis. The computation begins by setting all the $p_{k,m}$ values to ambient pressure p_a. The mesh is traversed until convergence is obtained, upon which control then passes to the integrating routine for calculation of load and center of pressure. If the latter does not fall on the pivot axis (within preset limits), the angle is automatically adjusted while the minimum clearance is held fixed. A new solution to the difference equation is obtained using the previous solution as a starting approximation. When a sufficiently precise angle is found, the computer punches a card containing the load, correct angle, and other information.

The computer program detects instability and automatically takes the corrective measures described in the preceding section. Weak instability is indicated by failure of the convergence indicator (8) to decrease from one traverse of the mesh to the next, while strong instability is detected by an overflow condition resulting from excessively large $p_{k,m}$ values. The corrective measures are specified by a table stored in memory. Each table word specifies a fraction by which γ is to be reduced and a fraction by which $\Delta x/\Delta y$ is to be increased. The program tries these remedies in sequence until either a stable condition is achieved or until the end of the table is reached. In the latter event the machine stops because any further corrective measures would result in excessively slow convergence.

The program also allows load-angle calculations to be performed for a sequence of values of either minimum clearance or surface curvature. Since the speed of calculation depends heavily upon the accuracy of the initial guess for the angle, considerable time is saved by extrapolating the previous correct angles to obtain the initial guess. The extrapolation is effected by means of a Newtonian backward-difference formula.

Fixed decimal arithmetic was employed. Typical operating speeds are 5-10 minutes to obtain convergence on a 66-point mesh, 20 minutes to find an equilibrium angle.

Figure 2 **Center of pressure plotted against $(\Delta x)^2$ for a cylindrical surface bearing.** *(Breadth = 0.6 in., length = 0.55 in., crown height = 250 μin.; velocity = 2500 ips; min. film thickness = 250 μin.; viscosity = 2.76 × 10⁻⁹ lb. sec./in.²; angle = 1.5 × 10⁻³ radians; $\Delta x/\Delta y = 1.091$.)*



Reference

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Received June 18, 1958