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Noise Theory for Hot Electrons

The subject of this note is the extension of the Nyquist theorem, which relates the electrical noise in a conductor in thermal equilibrium to the Ohmic conductivity,[1,2] to the "hot electron" situation in which there is a steady electric field **E** strong enough to disturb the distribution of electrons among Bloch states and for which the steady current density **J** will not in general be proportional to **E**.[3] For simplicity, we discuss the fluctuations of velocity, **v**, for a *single electron*. For a steady state the *time* average,

$$\bar{\psi} \equiv \underset{s\to\infty}{\mathrm{Lim}} \left(\frac{1}{s} \int_0^s \psi(t)\,dt \right),$$

of a function ψ of the electron's state is the same as the *ensemble* average at any instant over all N electrons in the body:

$$\bar{\psi} = If\psi \equiv \langle\psi\rangle \,.$$

(I stands for integration over the Brillouin zone and summation over spin states and bands, f is the distribution function normalized so that $If=1$.) For fluctuations we have (neglecting any interaction between the electrons)

$$\overline{(\Sigma_{i=1}^{N}\psi_i - N\bar{\psi})^2} = N\overline{(\psi-\bar{\psi})^2} \,,$$

and similarly for each spectral component of the fluctuation.[4] (Here, and below wherever it is material, ψ is taken to be real.) We are interested in the "noise" spectral density

$$G(\psi',\omega) \equiv \underset{s\to\infty}{\mathrm{Lim}} \left(\frac{2}{s} \left| \int_0^s \psi'(t)\, e^{-i\omega t} dt \right|^2 \right), \tag{1}$$

where $\psi' \equiv \psi - \bar{\psi}$. This has the property

$$\int_0^\infty G\,d\omega = 2\pi \overline{(\psi-\bar{\psi})^2} \,. \tag{2}$$

The relation which links G to the transport coefficients of the system is the Weiner-Khinchin theorem[1]

$$G(\psi',\omega) = 4\,\mathrm{Re} \int_0^\infty e^{-i\omega s} \overline{(\psi'(t)\psi'(t+s))}\, ds \,. \tag{3}$$

The first factor, $\psi'(t)$, of the time average $\overline{(\quad)}$ in (3) may be replaced by $\psi(t)$ (since, obviously, if it were replaced by $\bar{\psi}$ the time average would become zero). For comparison with transport coefficients, it is convenient to formulate the right-hand side of (3) as follows: Let the expectation of ψ after an interval s, for an electron which started from a particular specified state at the initial time, be $\langle\psi; s\rangle$. This expectation will be a function of the initial state and of the steady fields (**E** and **H**) acting, but not of the initial time. Of course,

$$\langle\psi;\infty\rangle = \langle\psi\rangle \,.$$

(It should not be overlooked that, from their definitions, $\langle\psi; s\rangle$ is in general a function of electron state whereas $\langle\psi\rangle$ is a constant.) We may now write (3) as

$$G(\psi',\omega) = 4\,\mathrm{Re} \left\langle \psi \int_0^\infty e^{-i\omega s} (\langle\psi; s\rangle - \langle\psi\rangle)\, ds \right\rangle \,. \tag{4}$$

Now, the Ohmic mobility tensor for *thermal equilibrium* has diagonal elements[5,6]

$$\mu_{xx}(\omega) = \frac{e}{kT} \left\langle v_x \int_0^\infty e^{-i\omega s} \langle v_x; s\rangle\, ds \right\rangle \,. \tag{5}$$

Therefore, for this case, we have at once

$$4kT\,\mathrm{Re}\,\mu_{xx}(\omega) = eG(v_x,\omega) \,, \tag{6}$$

which is essentially the Nyquist relation for the fluctuation of the current. By combining (6) with the Einstein relation,

$$kT\boldsymbol{\mu}(0) = e\boldsymbol{\zeta} \,, \tag{7}$$

connecting $\boldsymbol{\mu}$ to the diffusion constant $\boldsymbol{\zeta}$ (for the electron flux due to a concentration gradient, with f constant), we obtain

$$G(v_x, 0) = 4\zeta_{xx} \,. \tag{8}$$

We now turn to the "hot electron" situation. Both G and $\boldsymbol{\zeta}$ are essentially *differential* magnitudes, referring here to deviations from the steady state condition — for the latter, which may be written as $\boldsymbol{\zeta}'$ to remind one of its meaning, specifically to small deviations linearly proportional to each other. On the other hand a distinction must now be made between the absolute mobility $\boldsymbol{\mu}$, given by $\langle\mathbf{v}\rangle \equiv \mathbf{u} = \pm\mathbf{E}\cdot\boldsymbol{\mu}$, and the differential mobility $\boldsymbol{\mu}' \equiv \pm(\partial/\partial\mathbf{E})\mathbf{u}$. By the methods and results of reference 6, we have

$$\boldsymbol{\zeta}' = \left\langle \mathbf{v} \int_0^\infty (\langle\mathbf{v}; s\rangle - \mathbf{u})\, ds \right\rangle \,. \tag{9}$$

By combining (4) with (9) we retrieve (8), in the form

$$G(v_x', 0) = 4\zeta'_{xx} \,, \tag{8'}$$

as an exact result. Accordingly, generalization of the

Nyquist formula for low frequencies[7] and generalization of the Einstein relation are equivalent questions. One wishes to define a "noise temperature," θ_N, such that when it is substituted for T the generalization of (6) holds. For any given direction, there are two natural definitions:

$$\tfrac{1}{4}eG(v_x', 0) \equiv k\theta_{Nx}\mu_{xx}(0) \equiv k\theta'_{Nx}\mu'_{xx}(0). \tag{10}$$

Because of (8′), each "diffusion temperature," defined in the corresponding way to generalize (7), is equal to the equivalent noise temperature. A natural definition of an "energy temperature," θ_E, for comparison is:

$$\tfrac{3}{2}k\theta_E \equiv \langle\varepsilon\rangle\,, \tag{11}$$

where ε is the electron energy (relative to a band edge). It will be convenient, however, with the band structure expressed by (13), to further define an energy temperature for each direction:

$$k\theta_{Ex} \equiv m^*\langle v_x^2\rangle\,, \tag{12}$$

We shall examine two simple hot-electron situations below, and find that corresponding noise and energy temperatures are—as one might hope—of the same order of magnitude, though not in general equal.

The first case is that considered in reference 6. The scattering is by weak non-polar interaction with longitudinal Debye modes, and

$$\varepsilon = p^2/2m^*\,. \tag{13}$$

($\mathbf{p}$ is the pseudomomentum, m^* a constant.) With increasing applied field E, u becomes proportional to $\sqrt{E}$ while the surfaces of constant f remain almost spherical:

$$f \sim f(\varepsilon) = \text{const.}\,e^{-\varepsilon^2/\text{const.}}\,. \tag{14}$$

Let the z axis be parallel to $\mathbf{E}$. From the spherical symmetry of the system (i.e., independently of the actual distribution function) it follows that

$$\mu'_{xx} = \mu_{zz}\,. \tag{15}$$

From the results of reference 6, we have[8]

$$\left\{\begin{aligned} \mu'_{xx}(0) &= \frac{2}{3}\frac{e}{m^*}\langle\tau\rangle\,, && (16)\\ \zeta'_{xx} &= \frac{2}{3}\frac{1}{m^*}\langle\tau\varepsilon\rangle\,. && (17)\end{aligned}\right.$$

The ratio of the right-hand sides of (16) and (17) gives a noise temperature according to (8′) and (10). For thermal equilibrium ($E=0$, f proportional to $\exp(-\varepsilon/kT)$), we find $\theta'_{Nx} = \theta_E = T$. In the "$\sqrt{E}$ range," where (14) holds, we get

$$\frac{\theta'_{Nx}}{\theta_E} = \frac{\tfrac{3}{4}!}{(\tfrac{1}{4}!)(\tfrac{1}{2}!)} = 1.144\,. \tag{18}$$

For the field direction we have,[6] instead of (17),

$$\zeta'_{zz} = \frac{2}{3}\frac{1}{m^*}\left\langle \tau\left(\varepsilon - W\frac{d(\tau_0\varepsilon)}{d\varepsilon}\right)\right\rangle\,, \tag{19}$$

where $W \equiv eu^2/\mu_{zz}$ is the rate, per electron, of absorption of power from the field and $\tau_0(\varepsilon)$ is the function introduced in Eq. (8) of reference 6. Since $\tau_0(\varepsilon)$ is positive and monotonic-increasing, we conclude that $G(v_z', 0) < G(v_x', 0)$.

The second hot-electron situation to be considered has been discussed in detail by Gunn (reference 3, pages 217-219). As in the first example, we have spherical symmetry (so (15) still holds), with the energy given by (13); but the predominant scattering process is supposed to be that in which an optical-mode phonon, with energy ε_0, is created. It is supposed that for almost all such scattering events the electron's initial energy is only slightly greater than ε_0 (and its final energy $\ll \varepsilon_0$), and that in the interval between them (while the electron's energy is being restored by acceleration) an *acoustic*-mode scattering is improbable.[9] The distribution represented by $f(\mathbf{p})$ is then concentrated uniformly along a thin filament extending, in the direction of $(-)\mathbf{E}$ (which we choose for the z direction), from $p=0$ to $p_z = \sqrt{(2m^*\varepsilon_0)} \equiv p_0$. The duration of a scattering-acceleration-scattering cycle is

$$\tau_0 = p_0/eE\,. \tag{20}$$

To this first approximation,*

$$\mathbf{u} = (0, 0, \tfrac{1}{2}v_0)\,;\ \mu_{zz} = \tfrac{1}{2}\tau_0 e/m^*\,;\ \mu'_{zz} = 0\,, \tag{21}$$

where $v_0 \equiv p_0/m^*$.

The diffusion constants (and hence the low-frequency noise powers) may be calculated in terms of the differential mean free path[6]

$$\mathbf{l}' \equiv \int_0^\infty (\langle\mathbf{v}; s\rangle - \mathbf{u})\,ds\,, \tag{22}$$

since, by (9), $\boldsymbol{\zeta}' = \langle\mathbf{v}\mathbf{l}'\rangle$. We have[10]

$$l'_x = v_x(p_0 - p_z)/eE\,,$$

and therefore

$$\zeta'_{xx} = \tfrac{1}{2}\tau_0\langle v_x^2\rangle\,. \tag{23}$$

This result (23), which we may regard as a good approximation so long as $\langle v_x^2\rangle \ll v_0^2$, gives

$$\theta'_{Nx} = \theta_{Ex} \tag{24}$$

by (8′), (10) and (12). The high-frequency noise may be calculated in the same way with a phase factor $\cos(\omega s)$ in the integrand of (22). We find

$$\frac{G(v_x', \omega)}{G(v_x', 0)} = \left(\frac{\sin\tfrac{1}{2}\omega\tau_0}{\tfrac{1}{2}\omega\tau_0}\right)^2. \tag{25}$$

It should be noted that this result is for the idealized limit corresponding to (21). One would expect the higher secondary maxima of the "diffraction pattern" (25) to be the most sensitive to departures from this limit, being quenched by the smearing out of the distribution of intervals between scatterings.

*See the remarks immediately following Eq. (28).

The analysis for the field direction is not so straightforward. From Eq. (7) of reference 6 we have, for the idealized limit,

$$\left.\begin{aligned} &v_0 dl'_z/dv_z+\tau_0(v_z-\tfrac{1}{2}v_0)=0\,, \qquad 0<v_z<v_0\,; \\ &\text{and} \\ &l'_z(v_z=0)=l'_z(v_z=v_0)\,. \end{aligned}\right\} \tag{26}$$

Therefore

$$l'_z=\tau_0 v_0\left(\frac{1}{2}\left(\frac{v_z}{v_0}\right)-\frac{1}{2}\left(\frac{v_z}{v_0}\right)^2-\frac{1}{12}\right), \tag{27}$$

and

$$\langle v_z l'_z\rangle=0\,. \tag{28}$$

Thus the low-frequency noise appears only from a closer approximation to the actual situation than is represented by (21), in which the "smearing out" of the distribution and of the electron paths is represented. It is known that $f(v_z)$ drops to zero (around $v_z=0$ and $v_z=v_0$) over a velocity range of order $\sqrt{\langle v_x^2\rangle}$, when the latter is $<<v_0$: presumably $\langle v_z l'_z\rangle$ is then of order $\tau_0\langle v_x^2\rangle$. That is, $G(v_x', 0)$ and $G(v_z', 0)$ are of the same order of magnitude. One would expect the high-frequency noise to have peaks at frequencies which are integer multiples of $1/\tau_0$, as in the transverse case above. The peaks for the longitudinal case presumably are much larger, however, since by (2)

$$\frac{\int_0^\infty G(v_z', \omega)d\omega}{\int_0^\infty G(v_x, \omega)d\omega}=\frac{\langle(v_z')^2\rangle}{\langle v_x^2\rangle}=\frac{v_0^2}{12\langle v_x^2\rangle}>>1\,.$$

A meaningful calculation of the high-frequency noise power, $G(v_z', \omega)$, evidently can be made only by going beyond the idealized extreme situation corresponding to (21).[11]

Footnotes and references

1. For a general reference, see C. Kittel, *Elementary Statistical Physics,* Sections 27-30 (Wiley, New York 1958).
2. For a treatment of the thermal-equilibrium case parallel with the analysis given here, see for example R. Kubo, *J. Phys. Soc. Japan* **12,** 570 (1957); M. Lax, *Phys. Rev.* **109,** 1921 (1958).
3. For a general reference on "hot electrons," see J. B. Gunn, *Progress in Semiconductors* **2,** 211 (Wiley, New York 1957).
4. Here we are assuming Boltzmann statistics to apply. Otherwise, for thermal equilibrium the correct result is obtained by replacing the sum-integral over the Fermi function f_0 (which of course differs from f as defined in the text in not having a normalizing constant) by that over $f_0(1-f_0)$. Because just the same substitution applies to the expression for the Ohmic mobility, the same Nyquist relation is valid for Fermi statistics as for Boltzmann statistics.
5. P. J. Price, *IBM Journal* **2,** 200 (1958).
6. P. J. Price, *Proceedings of the 1958 International Conference on Semiconductors, J. Phys. Chem. Solids* **8,** 136 (1959). The factor exp $(i\omega s)$ in the mobility formulas, for a harmonically varying applied field, is not included in this paper or in reference 5. The proof of this generalization is elementary, however.
7. For ω small compared with the frequencies, $1/\tau$, of electron relaxation collisions, $G(v_x', \omega)$ may be equated to $G(v_x', 0)$: the noise is "white" until the predominant collision frequencies are reached.
8. Both these results depend on the assumption that the part of $f(\mathbf{p})$ even in $\mathbf{p}$ is spherically symmetrical (a function of p only). In addition (17) depends, in deriving from Eq. (8) of reference 6, on the related circumstance that the expectation of change of ε, due to the accelerating field, between collisions is small compared to ε. The particular form of (16) given depends on our assumption that the scattering time $\tau(\varepsilon)$ is proportional to $1/\sqrt{\varepsilon}$.
9. These assumptions could be simultaneously valid only in a range of field E between finite upper and lower limits.
10. We are, of course, assuming here that the expectation of v_x after a scattering is zero.
11. One may make a formal calculation by adding a term $(i\omega\tau_0 l'_z)$ to the differential equation (26) and taking the real part of the solution (which is equivalent to inserting the phase factor $\cos(\omega s)$ in the integrand of (22)), and proceeding as before. The result is that $G(v_z', \omega)=0$ *at all frequencies.* The procedure is objectionable for frequencies near integral multiples of $1/\tau_0$, because the "generalized mean free path" obtained in this way has singularities at these points. The result does, however, seem to indicate that for all other frequencies, as for $\omega=0$, G is "zero" in the sense of being small compared with $\tau_0 v_0^2$.

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