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Numerical Solution of Laplace's Equation, Given Cauchy Conditions†

Usually Laplace's equation $\nabla^2\psi=0$ (or Poisson's equation $\nabla^2\psi=F$) must be solved with conditions given all around the boundary of the region in question. Yet in such specific engineering problems as the design of electron guns,^{1,2} solutions are sought in an open region with the Cauchy boundary condition.** With Cauchy conditions, Laplace's equation is "unstable"³ in that an exponential growth of errors occurs during numerical analysis by methods of finite differences. An expression that gives the order of magnitude of the propagated errors could therefore be of considerable value as a "rule of thumb" where these methods are used, particularly for digital computer programmers. This communication explains how this "rule of thumb" has been obtained.

The boundary C is, in general, not a straight line, nor can it be conformally mapped by analytic methods. Moreover, C may separate two different regions, respectively source-free (Laplace's equation) and source-present (Poisson's equation). When finite-difference methods are used, there are usually many "stars" which lie partially in each, no matter how fine the mesh is made. A typical five-point star,⁴⁻⁷ $ABCDE$ in Fig. 1, has points in both the the Laplace region and the Poisson region (shaded area). To start a solution over the Laplace region, the potential ψ at mesh points of the first four diagonal lines is calculated by Taylor-series expansion of ψ near the boundary C . The potential ψ and its normal derivative ψ_n are given along C ; partial derivatives of ψ along C are obtained by differentiating ψ , ψ_n along C and substituting in $\nabla^2\psi=0$ and its derivatives along and across C .⁸ Once ψ is known on several "starting" lines, the five-point star formula,

$$\psi_4 = 4\psi_0 - \psi_1 - \psi_2 - \psi_3,$$

can be used to calculate ψ at the rest of the mesh points to within the error of this difference approximation to the Laplacian operator. This technique of solving Laplace's equation by first determining several starting lines was originally suggested by Hyman.⁹

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†The practical aspect of this paper has been published as a note, "Numerical Analysis for Design of Electron Guns with Curved Electron Trajectories," *Proceedings of the IRE*, 47, 87-88 (January 1959).

**Added in proof: See F. John, "Notes on Improperly Posed Problems in Partial Differential Equations," Internal Report NN-117, Institute of Mathematical Sciences, New York University (August, 1958).

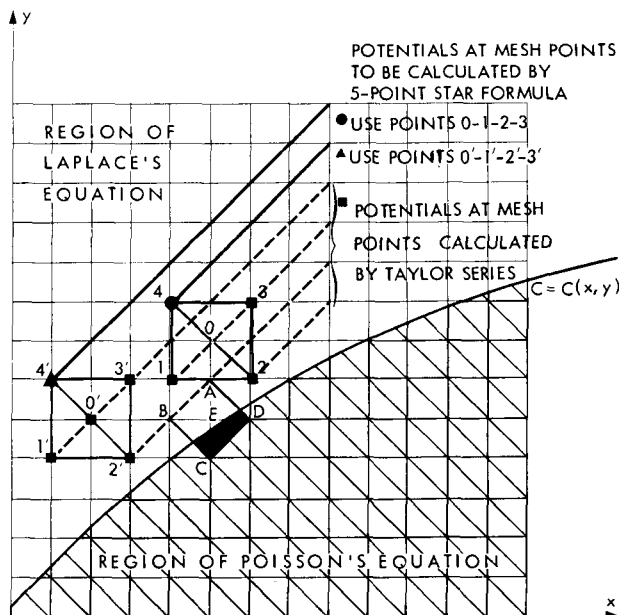


Figure 1 General boundary and five-point star.

Errors in the potentials at mesh points of the first four diagonal lines are inevitable because the Taylor-series expansions used to find them must be cut off after a finite number of terms, and the incremental distance for Taylor series (which is the mesh size) is small but never zero. The error at any point can be defined as

$$\varepsilon(x, y) = \psi_A(x, y) - \bar{\psi}(x, y), \quad (1)$$

where ψ_A is based on all the terms of the Taylor series and $\bar{\psi}$ on a finite number of terms. The implicit assumption of (1) is that when all the terms are used, the true potentials at mesh points in the vicinity of the boundary C can be obtained by Taylor-series expansions about potentials on C . Since even with multiple-precision arithmetic,¹⁰ existing digital computers will have a limiting error larger than the sum of the terms neglected, this error may be used for $\varepsilon(x, y)$.

The build-up of errors in an open region occurs as follows: Let $\varepsilon_{1,i}$ and $\varepsilon_{2,i}$ be the known errors at mesh

