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Two-Parameter Lifetime Distributions for Reliability Studies of Renewal Processes*

Abstract: **Probability functions are defined for use in reliability studies of equipments which are maintained over a long period of time through replacement of components. These are: *lifetime distribution function, lifetime density function, probability of survival, hazard, expected number of replacements,* and *renewal rate*. Theoretical results of renewal theory are adapted to reliability studies of complex systems.**

The "exponential law" is equivalent to the assumption that survival probability for any given time interval is independent of the age of a component at the beginning of the interval. It seems more realistic, however, to assume that this survival probability is a monotonically *decreasing* function of initial age, or, equivalently, that the hazard is a monotonically *increasing* function of the age of the component. Consequently, three two-parameter models of distribution functions, with the properties: (1) initial lifetime density greater than zero, and (2) monotonically increasing hazard, are proposed and discussed. The lifetime behavior associated with these models ranges from complete determinacy to complete randomness. An entropic measure of this randomness is introduced.

The expected number of replacements is numerically calculated and plotted as a function of time for several different parameter values in each model.

I. Introduction

Probabilistic methods used in reliability studies of equipments which are maintained over a period of time through replacement of components are examined in this paper. We propose several mathematical models to which statistical data obtained in studies of this kind may be fitted.

It is assumed that an equipment consists of many components, all needed for correct operation of the system. A component may be removed either because it has failed in operation or because its removal is prescribed by a maintenance procedure. Upon removal, it is immediately replaced by a new and statistically identical component.

In a reliability study, a record of these removals is kept for a period of time. The statistical data collected are used to determine the parameters in an analytic expression designed to approximate the probability of removal as a function of time. This expression is applied in predicting the behavior of this equipment and other similar systems.

*Presented at the NYU Industry Conference on Reliability Theory on June 9, 1958.

There are several probability functions useful for this type of analysis. They are all mathematically related to each other in such a way that any one uniquely determines all the others. The precise mathematical definition of these functions is given in Section II. At this point, we present an intuitive discussion of the statistical properties of equipment behavior which they represent.

Given a large number of components, new at time zero, which constitute the original population, the following functions will be approximated by the ratios they represent:

$F(t)$, the distribution function of lifetimes, represents that fraction of the original population which has been removed by time t.

$R(t)$, the probability of survival to age t, represents that fraction of the original population which is still operating at time t. Clearly, $R(t)=1-F(t)$. If components are removed only when they fail, this function is identical with component reliability, as the term is commonly used.

$f(t)$, the lifetime density function, represents the number of members of the original population being removed per unit time at time t divided by the entire original population. Thus $f(t)=dF(t)/dt$.

$z(t)$, the hazard, represents the number of members of the original population being removed per unit time at time t divided by the number which are still operating at time t. Thus $z(t)=f(t)/R(t)$. Hazard is often referred to as "force of mortality."

$U(t)$, the expected number of replacements, represents the total number of removals (from the original population plus later generations) up to time t, divided by the original population.

$u(t)$, the renewal rate, represents the total number of components (from the original population plus later generations) being removed per unit time at time t, divided by the original population. Thus $u(t)=dU(t)/dt$.

F, R, f, and z essentially describe the underlying behavior of a *component* as a function of its age. On the other hand, U and u are the functions which describe the behavior of an *equipment* containing many components in a renewal process.

It should be emphasized that the distribution function $F(t)$ defined below is dependent not only on the physical characteristics of the components, but also on the way in which they are used in the specific equipment, the environmental strains to which they are subjected, and, to a large degree, on the criteria for their removal.

These removal criteria are determined by the maintenance procedures which are prescribed for the equipment, and vary widely between two extremes. At one extreme a component is replaced only after it has caused the system to fail, while at the other, blocks of components are replaced simultaneously on a fixed schedule or whenever the maintenance engineer deems it advisable. Between these two extremes, diagnostic procedures have been developed, designed to find components which are likely to fail within a short time and to replace them before they cause the system to fail.

Another important factor in determining the distribution function is the choice of a particular measure of time. This choice will depend on the application for which the theory is to be used and may, for example, be calendar time, power-on time, or operating time.

Thus, our $F(t)$ is a distribution function of lifetimes of a specified type of component in a specified system under specified conditions of operation and maintenance.

II. Mathematical definition of functions used to characterize renewal processes

Consider a system maintained continuously in operating condition. Whenever a component is removed, it is immediately replaced by a new and statistically identical component in good condition. The time required for this replacement is assumed negligible. *Removal may be due to failure or to preventive maintenance procedure.* It is assumed that the system is observed for a certain period of its operating life, which we will call *the observation period.* The components in the system at the beginning of this observation period will have some age distribution.

In the following, upper case letters denote random variables and lower case letters denote the numerical values which they may assume. Let:

t be time measured from the beginning of the observation period,

x be time measured backwards from the beginning of the observation period,

T be the time-to-removal of a specific component in the system, measured from the beginning of the observation period,

X be the age of a specific component at time $t=0$, i.e. the time the component has been in the system prior to the start of the observation period,

$L = T + X =$ the total lifetime of the component in the system, i.e., the time-to-removal measured from the time the component was first put into the system.

a) *The distribution function of lifetimes* in a given application of a component of age zero at time $t=0$, is $F(t)$, defined as the probability that a component is removed by time t.

$$\begin{aligned} F(t)&=Pr\{L\leq t \mid X=0\}\ , && t\geq 0\ ; \\ F(t)&=0\ , && t<0\ , \end{aligned} \qquad \text{(II.1)}$$

where $Pr\{A|B\}$ denotes "the probability of A, conditional on B."

Our discussion will be limited to functions $F(t)$ which have the following properties:

i) $F(0)=0$;

ii) $F(t)$ is continuous, $-\infty<t<\infty$;

iii) $F'(t)$ exists and is continuous everywhere except at $t=0$;

iv) $\lim_{t\to\infty} F(t)=1$.

v) $\int_0^\infty t\,dF(t)$ exists.

b) *The lifetime density function* is $f(t)$:

$$\begin{aligned} &f(t)=F'(t)\ , \qquad t>0 \ \text{ or } \ t<0\ ; \\ &f(0)=F'(0+) = \text{the right derivative of } F(t) \text{ at } t=0\ . \end{aligned} \qquad \text{(II.2)}$$

c) *The probability of survival* to age t is $R(t)$:

$$R(t)=1-F(t)\ . \qquad \text{(II.3)}$$

d) *The distribution function of times-to-removal* for a component of age x at time $t=0$ is defined as the probability that a component of initial age $X=x$ is removed by time t. This is the truncated distribution:[11]

$$F(t;x)=Pr\{T\leq t \mid T>0,\ X=x\}$$

For $x\geq 0$, $F(x)<1$,

$$F(t;x)=\frac{F(x+t)-F(x)}{1-F(x)}\ , \qquad \text{for} \quad t\geq 0 \qquad \text{(II.4)}$$

and

$$F(t;x)=0\ , \qquad \text{for} \quad t<0\ .$$

Clearly

$F(t;0)=F(t)$.

e) The corresponding *removal density function*, $f(t;x)$, is defined as:

$$f(t;x)=\frac{dF(t;x)}{dt}=\frac{f(x+t)}{1-F(x)} \ . \quad t\geq 0 \ , x\geq 0 \ , F(x)<1 \ , \qquad \text{(II.5)}$$
$$=0 \ , \qquad t<0 \ , x\geq 0 \ , F(x)<1 \ .$$

f) *The probability of survival*, $R(t;x)$, of a component of age x at time $t=0$, is the probability that it is not removed during the time interval 0 to t.

$$R(t;x)=1-F(t;x)=\frac{R(x+t)}{R(x)} \ , \ t\geq 0 \ , x\geq 0 \ , F(x)<1 \ , \qquad \text{(II.6)}$$
$$=1 \ , \qquad t<0 \ , x\geq 0 \ , F(x)<1 \ .$$

g) *Hazard:*

i) Consider a component of age 0 at $t=0$ which is known to survive to time t. The truncated function $F(\Delta t;t)$ is the distribution function of times to removal Δt, with origin at time t.

$$F(\Delta t;t)=\frac{F(t+\Delta t)-F(t)}{1-F(t)} \ , \qquad \Delta t\geq 0 \ , \quad F(t)<1 \ .$$

Then the hazard $z(t)$ is defined as:

$$z(t)=\lim_{\Delta t\to 0+}\left(\frac{1}{\Delta t}F(\Delta t;t)\right)=f(0;t)$$
$$=\frac{f(t)}{1-F(t)} \ , \quad t\geq 0 \ , F(t)<1 \ . \qquad \text{(II.7)}$$

ii) For a component of age x at $t=0$, which is known to survive to time t or age $x+t$,

$$z(t;x)=\lim_{\Delta t\to 0+}\left(\frac{1}{\Delta t}F(\Delta t;x+t)\right)$$
$$=\frac{f(x+t)}{1-F(x+t)} \ , \qquad t\geq 0, x\geq 0 \ , \ F(x+t)<1 \ . \qquad \text{(II.8)}$$

Thus,

$$z(t;x)=z(x+t) \ , \qquad \text{(II.9)}$$

or, in other words, $z(t;x)$ is a function only of the total age of the component.

A useful relation between hazard, $z(t;x)$, and survival probability, $R(t;x)$, may be derived as follows, making use of (II.6):

$$z(t;x)=z(x+t)=-\frac{\partial}{\partial t}\ln R(x+t)$$
$$=-\frac{\partial}{\partial t}\ln R(t;x)$$

Therefore

$$R(t;x)=\exp\left[-\int_x^{x+t} z(u)du\right] . \qquad \text{(II.10)}$$

h) *The mean lifetime*, t_m, of a component in the system, i.e. the expected time-to-removal measured from the time the component is put into the system, is[1]

$$t_m=\int_0^\infty tf(t)dt=\int_0^\infty R(t)dt \ . \qquad \text{(II.11)}$$

i) When considering a component in a system which is immediately replaced by a new and statistically identical component when it is removed, two quantities are of interest and are defined as follows:

Let $N(t;x)$ be the number of replacements up to time t, where the initial component has age x at $t=0$. $N(t;x)$ is a random variable for any t and x. Then the *expected number of replacements* by time t, $U(t;x)$, is defined as

$$U(t;x)=E\left\{N(t;x)\right\} \ , \qquad \text{(II.12)}$$

and the renewal rate is $u(t;x)$, where

$$u(t;x)=\frac{\partial U(t;x)}{\partial t} \qquad \text{(II.13)}$$

The quantities $U(t;x)$ and $u(t;x)$ may be expressed in terms of the previously defined functions as follows:

Given an initial component with a known age $X=x$ at time $t=0$, and considering each component to have the same lifetime distribution function, $F(t)$, the distribution for the ith generation is

$$U_i(t;x)=Pr\left\{T_1+L_2+\cdots+L_i\leq t\right\} \ , \qquad \text{(II.14)}$$

where T_1 is the time-to-removal of the first component, L_2, $L_3\cdots$ are the operating lifetimes for the second, third, $\cdots$ components respectively. The distribution function of the sum of i random variables is obtained by $i-1$ convolutions,[16] so that

$$U_1(t;x)=F(t;x) \ ,$$
$$U_2(t;x)=\int_0^t U_1(t-\tau;x)f(\tau)d\tau$$
$$\vdots \qquad \text{(II.15)}$$
$$U_i(t;x)=\int_0^t U_{i-1}(t-\tau;x)f(\tau)d\tau \ .$$

Then it may be shown[2] that

$$U(t;x)=E\left\{N(t;x)\right\}=\sum_{i=1}^{\infty}U_i(t;x) \ . \qquad \text{(II.16)}$$

From (II.15) and (II.16) one may derive an integral equation for $U(t;x)$:

$$U(t;x)=U_1(t;x)+\int_0^t\left\{\sum_{i=1}^{\infty}U_i(t-\tau;x)\right\}f(\tau)d\tau \ ,$$

and therefore

$$U(t;x)=F(t;x)+\int_0^t U(t-\tau;x)f(\tau)d\tau \ . \qquad \text{(II.17)}$$

By (II.13), and since $U(0;x)=0$,

$$u(t;x)=f(t;x)+\int_0^t u(t-\tau;x)f(\tau)d\tau \ . \tag{II.18}$$

In the special case where the initial component has age 0 at $t=0$, we let

$$U(t)\equiv U(t;0)=F(t)+\int_0^t U(t-\tau)f(\tau)d\tau \tag{II.19}$$

$$u(t)\equiv u(t;0)=f(t)+\int_0^t u(t-\tau)f(\tau)d\tau \ . \tag{II.20}$$

In general the age, X, of the initial component at the beginning of an observation period may not be precisely known. However, if its distribution function $\Phi(x)$ is known, F may be generalized as follows:

$$F_\Phi(t)=\int_0^\infty F(t;x)d\Phi(x) \ . \tag{II.21}$$

Then, the integral equation determining U may be generalized to:

$$U_\Phi(t)=F_\Phi(t)+\int_0^t U_\Phi(t-\tau)f(\tau)d\tau \ . \tag{II.22}$$

These integral equations, (II.19), (II.20) and (II.22), sometimes known as the renewal equations, have been extensively studied both in the theory of integral equations and in the theory of renewal processes.[2,3,4,5,7] The following results applicable to this study of systems maintained by renewal processes have been obtained:

a) Since $F(t)$ and $F_\Phi(t)$ are finite, non-decreasing, continuous functions for $0\leq t<\infty$, with $F(0)=0$, $F_\Phi(0)=0$, there exists, for all $t\geq 0$, a unique, non-decreasing continuous function $U(t)$ or $U_\Phi(t)$ satisfying (II.19) or (II.22).

b) Since $f(t)$ is continuous and non-negative for $0<t<\infty$, and right-continuous at $t=0$, there exists, for all $t\geq 0$, a unique, non-negative, continuous function $u(t)$ satisfying (II.20).

c) Since $f(t)$ is continuous and finite for $0<t<\infty$, $\int_0^\infty f(t)dt=1$, and $t_m=\int_0^\infty tf(t)dt$ exists, then

$$\lim_{t\to\infty} u(t)=\frac{1}{t_m} \ ; \tag{II.23}$$

i.e. as t approaches infinity, the renewal rate approaches a constant value. This implies

$$\lim_{t\to\infty}\frac{U(t+h)-U(t)}{h}=\frac{1}{t_m}, \text{ for all } h,$$

and (II.24)

$$\lim_{t\to\infty}\frac{U(t)}{t}=\lim_{t\to\infty}\frac{1}{t}\int_0^t u(\tau)d\tau=\frac{1}{t_m} \ .$$

These results have been proved by Feller[3] using Laplace transform methods, and by Doob[2] using probabilistic methods.

d) For $t=0$, the lifetime density function, $f(t)$, the hazard, $z(t)$, and the renewal rate, $u(t)$, are all equal:

$$f(0)=z(0)=u(0) \ . \tag{II.25}$$

Moreover, if the right derivative of $f(t)$ at $t=0$ exists, by differentiating (II.7) and (II.20), it may be seen that at $t=0$ the slopes of the hazard, $z(t)$, and renewal rate, $u(t)$, are equal:

$$\left.\frac{dz}{dt}\right|_{t=0+}=\left.\frac{du}{dt}\right|_{t=0+}=\left.\frac{df}{dt}\right|_{t=0+}+[f(0)]^2 \ . \tag{II.26}$$

III. Exponential law

A special case of interest is that in which the probability of survival of a component for any interval 0 to t is independent of the age of the component at the beginning of the interval:

$$R(t;x)\equiv\frac{R(t+x)}{R(x)}=R(t) \ , \qquad \text{for all } t \text{ and } x>0 \ .$$

Therefore

$$R(t+x)=R(t)R(x) \ . \tag{III.1}$$

The only continuous function which will satisfy these conditions is an exponential function.[17] Since $\lim_{t\to\infty} R(t)=0$, we have

$$R(t)=e^{-\lambda t}, \qquad \lambda>0 \ . \tag{III.2}$$

Then

$$F(t;x)=F(t)=1-e^{-\lambda t} \ , \qquad t\geq 0$$

$$f(t;x)=f(t)=\lambda e^{-\lambda t} \ , \qquad t\geq 0$$

$$t_m=\int_0^\infty e^{-\lambda t}dt=\frac{1}{\lambda} \ , \tag{III.3}$$

$$z(t)=\frac{f(t)}{R(t)}=\lambda \ . \qquad t\geq 0 \tag{III.4}$$

Also, it is shown in Appendix A that for this case

$$U(t;x)=\lambda t \ , \qquad t\geq 0 \tag{III.5}$$

and

$$u(t;x)=\lambda \ , \qquad t\geq 0 \ . \tag{III.6}$$

This distribution function is the model for the well-known "exponential failure law." For this distribution, and only for this distribution, the hazard and renewal rate are equal for all $t\geq 0$, and have a constant value equal to the reciprocal of the mean lifetime.

$$u(t)=z(t)=1/t_m=\lambda \ , \qquad 0\leq t<\infty \ . \tag{III.7}$$

The exponential law corresponds to maximum "randomness" of lifetimes. This idea will be precisely defined in Section VII.

IV. Proposed models

The lifetime distribution, $F(t)$, of a component in a complex system is generally an unknown function which depends on the physical properties of the component and its environment. Available information about $F(t)$ may be classified into two categories: first, failure data observed in the system which provide a statistical image of $F(t)$, and second, knowledge of the physical causes of failure. In order to utilize this information in reliability studies of systems, mathematical models of $F(t)$ are postulated. It is required that the mathematical form of these models be plausible in the light of existing knowledge of the physics of failure, and that they have undetermined parameters which may be estimated from available statistical data.

The basic assumption underlying the exponential model is that the probability of survival of a component for any given time interval is independent of its age at the beginning of the interval. However, in most practical situations it seems far more reasonable to assume that this survival probability will decrease with increasing initial age. (We rule out "infant mortality" on the assumption that adequate acceptance testing will eliminate this effect.) In Appendix B, it is shown that the assumption that survival probability, $R(t;x)$, decreases with increasing initial age x for any given time interval, t, is equivalent to the assumption that hazard is a monotonically increasing function of the age of the component. All the models considered in the following have this property.

The second important property common to all the models we consider is:

$$z(0)=f(0)=u(0)>0 \ ,$$

i.e. the initial hazard for a new component is non-zero. This assumption is justified by the fact that, in many reliability studies, large quantities of data which lead to definite positive values for $z(0)$ have been obtained.[10,12] In general, more information is available about the value of the lifetime density near zero than at any other time. Therefore, the value of $f(0)$ is an important quantity in the selection of an appropriate distribution to describe the lifetime behavior of a class of components.

Several models in which hazard increases with age have been proposed in the past. Among these are the normal, the log normal, the gamma, and the Weibull distributions.[8,9,10] For each of these models, $z(0)$ is zero. For all the models considered below, $z(0)$ may assume any prescribed positive value, so that the required flexibility in the choice of a model is achieved.

At least two parameters are needed to fit a model with the above properties to observed data. However, it is felt that the quality of lifetime data which can be obtained in most practical situations does not justify the use of more than two parameters. The following discussion is therefore limited to two-parameter distribution functions.

Three models have been considered and will be denoted by the superscripts (1), (2), and (3). $F^{(1)}$ and $F^{(2)}$ were derived from assumptions of linear and quadratic dependence of hazard on the age of a component. $F^{(3)}$ is a "truncated normal" distribution and will be discussed in detail in Section V.

Let

$$z^{(1)}(t)=a+2b^2t \ , \quad \text{where} \quad 0<a<\infty, 0<b<\infty \ , \qquad \text{(IV.1)}$$

$$z^{(2)}(t)=a+3c^3t^2 \ , \quad \text{where} \quad 0<a<\infty, 0<c<\infty \ . \qquad \text{(IV.2)}$$

By (II.10) for $x=0$,

$$F^{(1)}(t)=1-R^{(1)}(t)=1-\exp\left[-(at+b^2t^2)\right] \ , \qquad \text{(IV.3)}$$

$$F^{(2)}(t)=1-R^{(2)}(t)=1-\exp\left[-(at+c^3t^3)\right] \ . \qquad \text{(IV.4)}$$

Similarly

$$f^{(1)}(t)=(a+2b^2t)\exp\left[-(at+b^2t^2)\right] \ , \qquad \text{(IV.5)}$$

$$f^{(2)}(t)=(a+3c^3t^2)\exp\left[-(at+c^3t^3)\right] \ , \qquad \text{(IV.6)}$$

$$R^{(1)}(t;x)=\exp\left[-(at+b^2t^2+2b^2tx)\right]$$
$$=R^{(1)}(t)\exp(-2b^2tx) \ , \qquad \text{(IV.7)}$$

$$R^{(2)}(t;x)=\exp\left[-(at+c^3t^3+3c^3t^2x+3c^3tx^2)\right]$$
$$=R^{(2)}(t)\exp\left[-3c^3tx(t+x)\right] \ . \qquad \text{(IV.8)}$$

For the mean life of a component, we have

$$t_m^{(1)}=\int_0^\infty R^{(1)}(t)dt=\frac{1}{b}\int_0^\infty \exp\left[-(y^2+\frac{a}{b}y)\right]dy \ .$$

$$t_m^{(1)}=\frac{\sqrt{\pi}}{b}\exp\left(\frac{1}{4}\frac{a^2}{b^2}\right)P\left(\frac{a}{\sqrt{2}b}\right) \ , \qquad \text{(IV.9)}$$

where

$$P(u)=\frac{1}{\sqrt{2\pi}}\int_u^\infty \text{xp}\left(-\frac{1}{2}\xi^2\right)d\xi \ .$$

$$t_m^{(2)}=\int_0^\infty R^{(2)}(t)dt=\frac{1}{c}\int_0^\infty \exp\left[-(y^3+\frac{a}{c}y)\right]dy \ . \qquad \text{(IV.10)}$$

This integral may be evaluated numerically for any value of a/c. For the limiting case of $a=0$,

$$t_m^{(2)}=\frac{1}{3c}\Gamma\left(\frac{1}{3}\right)=\frac{0.893}{c} \ .$$

In both these models, the initial hazard, $z(0)$, is represented by the parameter, a, and the rate of increase of the hazard with age is determined by parameter b or c. Thus, as b or c approaches zero, with a finite, either of these distributions approaches the exponential. Conversely, as a approaches zero, with b or c finite either function approaches a Weibull distribution.[8]

These models may be reparameterized in a manner which sheds additional light on their properties. One parameter, t_m, represents the mean life of a component and the other determines the shapes of the probabilistic functions on a time scale normalized in terms of the mean life. The ratios a/b and a/c determine the functional dependence of hazard, survival probability, and the lifetime density function on the normalized variable, $\theta=t/t_m$. For example, given

$$F^{(1)}(t)=1-\exp\left[-(at+b^2t^2)\right] \ , \qquad b>0 \ ,$$

with

$$t_m^{(1)}=\frac{1}{b}\int_0^\infty \exp\left[-(y^2+\frac{a}{b}y)\right]dy=\frac{1}{b}\phi^{(1)}\left(\frac{a}{b}\right) \ , \qquad \text{(IV.11)}$$

the corresponding function† of the normalized variable is $F^{*(1)}(\theta)$:

$$F^{*(1)}(\theta)=1-\exp\,[-(\alpha\theta+\beta^2\theta^2)] \tag{IV.12}$$

determined by the conditions

$$\theta_m=\frac{1}{\beta}\phi^{(1)}\left(\frac{\alpha}{\beta}\right)=1,\quad \frac{\alpha}{\beta}=\frac{a}{b}\ . \tag{IV.13}$$

Thus,

$$\beta=\phi^{(1)}(a/b)$$

and

$$\alpha=(a/b)\phi^{(1)}(a/b)\ . \tag{IV.14}$$

For the limiting case of $b=0$, with a finite,

$$F^{*(1)}(\theta)=1-e^{-\theta}\ ,$$

so that $\alpha=1$.

Correspondingly, for

$$F^{(2)}(t)=1-\exp\,[-(at+c^3t^3)]\ ,\qquad c>0\ ,$$

we have

$$F^{*(2)}(\theta)=1-\exp\,[-(\alpha\theta+\gamma^3\theta^3)]\ , \tag{IV.15}$$

where

$$\gamma=\phi^{(2)}\left(\frac{a}{c}\right)=\int_0^\infty \exp\left[-(y^3+\frac{a}{c}y)\right]dy\ , \tag{IV.16}$$

and

$$\alpha=(a/c)\phi^{(2)}(a/c)\ .$$

For the limiting case of $c=0$, with a finite,

$F^{*(2)}(\theta)=1-e^{-\theta}$, and $\alpha=1$.

The values of the parameters α, β, and γ for various values of a/b and a/c are given in Table 1.

In Figs. 1 to 6, $z^{*(1)}$, $z^{*(2)}$, $R^{*(1)}$, $R^{*(2)}$, $f^{*(1)}$, and $f^{*(2)}$ are plotted as functions of the normalized variable θ. These curves indicate the range of behavior which may be described by models of this form.

V. Truncated normal distribution

The normal distribution, for which

$$f(t)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right]\ ,$$

has been widely used in reliability studies, and a careful analysis of its properties as applied to the present discussion yields interesting results. It must be borne in mind that the normal distribution extends from $-\infty$ to $+\infty$, while lifetimes of components are limited to positive values.

† In what follows, the asterisk indicates that function of the normalized variable θ obtained from the corresponding function of t, by change of scale with $\theta=t/t_m$. For example:
$R^*(\theta)=R(t)$,
$f^*(\theta)=\frac{d}{d\theta}F^*(\theta)=t_mf(t)$.

Thus, one is led to use a distribution which is truncated at $t=0$.

$$\begin{aligned} f^{(3)}(t)&=0\ , \qquad t<0\ ;\\ &=\frac{1}{P(-t_0/\sigma)}\cdot\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right]\ ,\ t\geq 0 \end{aligned} \tag{V.1}$$

where

$$P\left(-\frac{t_0}{\sigma}\right)=\frac{1}{\sqrt{2\pi}}\int_{-t_0/\sigma}^{\infty}\exp\left(-\frac{1}{2}y^2\right)dy\ .$$

Then

$$R^{(3)}(t)=P\left(\frac{t-t_0}{\sigma}\right)\Big/P\left(\frac{-t_0}{\sigma}\right) \tag{V.2}$$

$$z^{(3)}(t)=\frac{f^{(3)}(t)}{R^{(3)}(t)}=\frac{1}{P\left(\frac{t-t_0}{\sigma}\right)}\cdot\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right], \tag{V.3}$$

$$R^{(3)}(t;x)=P\left(\frac{t+x-t_0}{\sigma}\right)\Big/P\left(\frac{x-t_0}{\sigma}\right)\ . \tag{V.4}$$

Figure 7 (at top of page 66) indicates the relative position of $t=0$ on the normal curve corresponding to various values of t_0/σ. Let us consider the behavior of this truncated distribution as t_0/σ ranges from $-\infty$ to $+\infty$. For t_0/σ large, say >3, it differs only slightly from the normal. As $\sigma\to 0$, with $t_0>0$, $F^{(3)}(t)$ approaches a step function with the limiting case

$$\begin{aligned} \lim_{\sigma\to 0} F^{(3)}(t)&=0\ , \qquad t<t_0\ ;\\ &=1 \qquad t\geq t_0\ . \end{aligned} \tag{V.5}$$

This corresponds to a fixed lifetime equal to t_0.

Table 1

$F^{(1)}$			
$\frac{\alpha}{\beta}=\frac{a}{b}$	α	β	H†
0	0	$\frac{1}{2}\sqrt{\pi}=0.886$	0.717
0.4	0.287	0.717	0.854
1.2	0.604	0.503	0.950
∞	1	0	1

$F^{(2)}$			
$\frac{\alpha}{\gamma}=\frac{a}{c}$	α	γ	H†
0	0	$\frac{1}{3}\Gamma\left(\frac{1}{3}\right)=0.893$	0.399
0.5	0.353	0.705	0.733
2.0	0.798	0.399	0.954
∞	1	0	1

†This quantity is defined in Section VII.

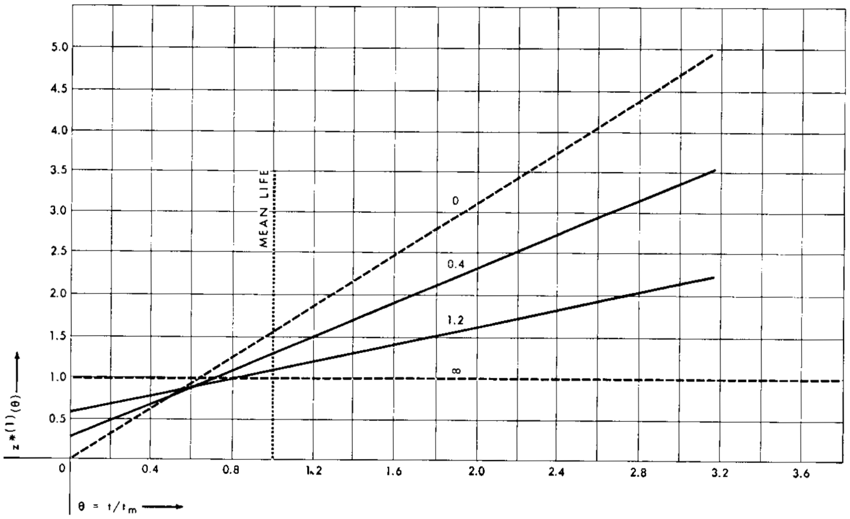
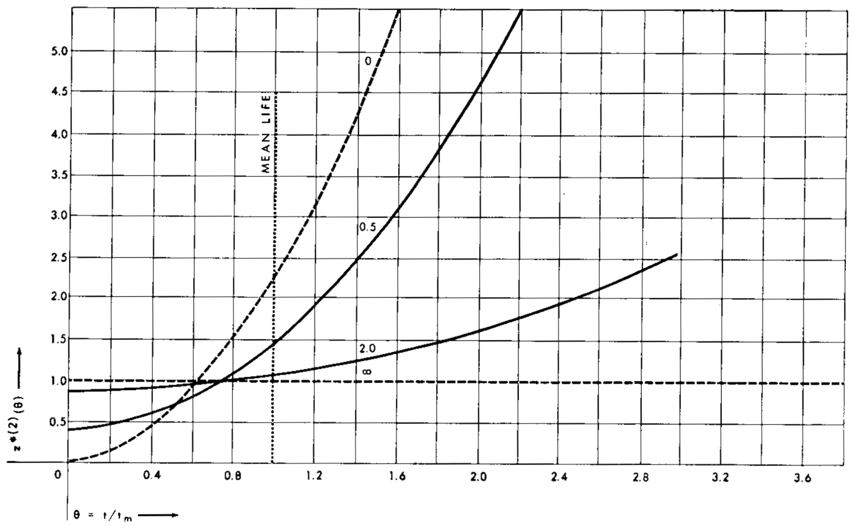


Figure 1
Hazard $z^{*(1)}(\theta) = \alpha + 2\beta^2\theta$.
Values of α/β *indicated on curves.*

Figure 2
Hazard $z^{*(2)}(\theta) = \alpha + 3\gamma^3\theta^2$.
Values of α/γ *indicated on curves.*



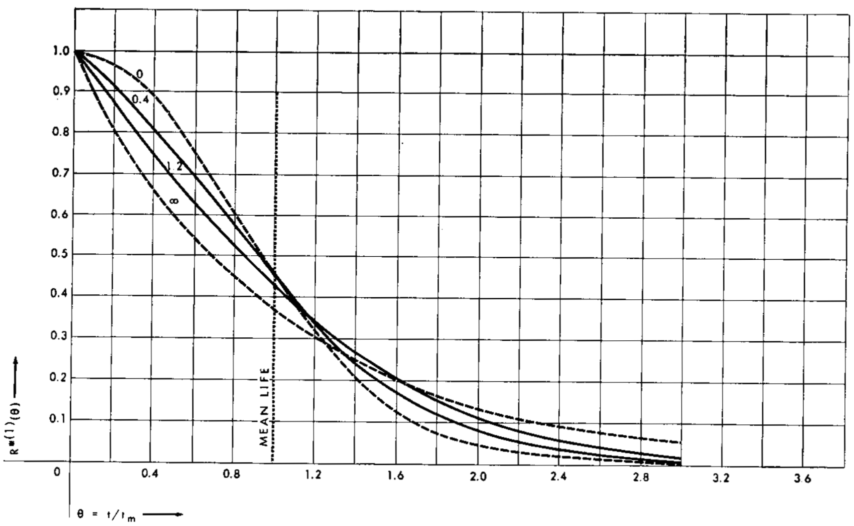


Figure 3
Survival probability
$R^{*(1)}(\theta) = \exp\ [-(\alpha\theta + \beta^2\theta^2)]$.
Values of α/β *indicated on curves.*

Figure 4
Survival probability
$R^{*(2)}(\theta) = \exp\left[-(\alpha\theta + \gamma^3\theta^3)\right]$.
Values of α/γ indicated on curves.

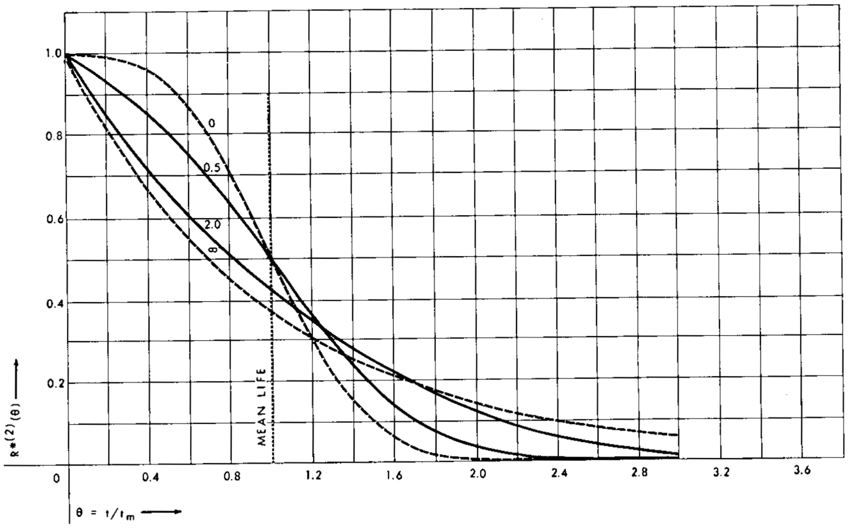


Figure 5
Lifetime density
$f^{*(1)}(\theta) = (\alpha + 2\beta^2\theta^2) \exp\left[-(\alpha + \beta^2\theta^2)\right]$.
Values of α/β indicated on curves.

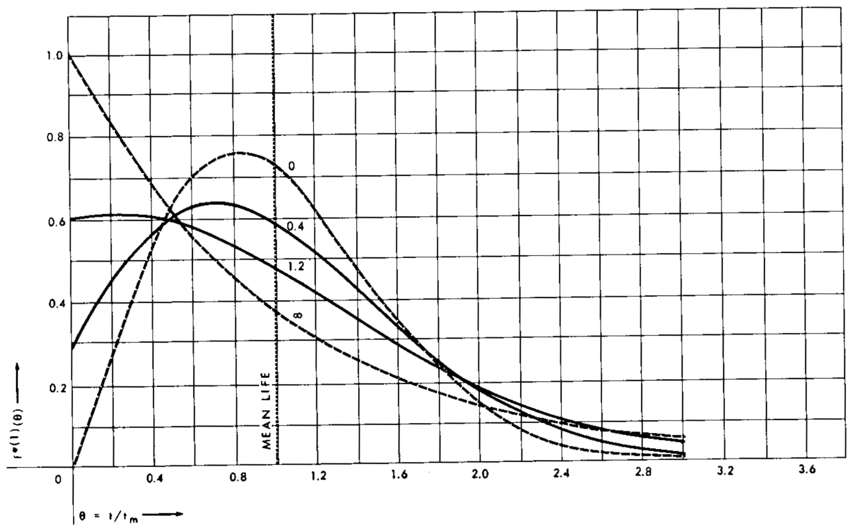
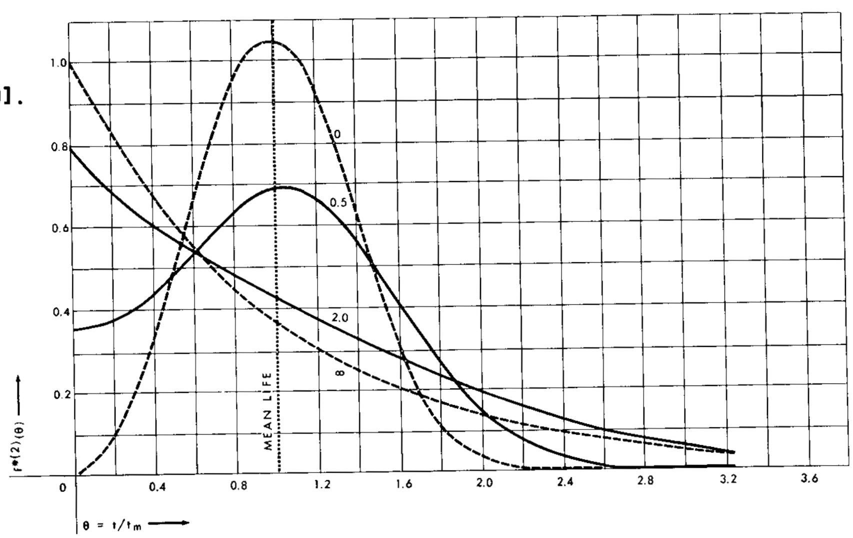


Figure 6
Lifetime density
$f^{*(2)}(\theta) = (\alpha + 3\gamma^3\theta^3) \exp\left[-(\alpha + \gamma^3\theta^3)\right]$.
Values of α/γ indicated on curves.



As $t_0/\sigma \to -\infty$, such that $t_0/\sigma^2 \to -\lambda$ where λ is a positive constant, the truncated normal distribution approaches the exponential:

$$\lim_{\substack{t_0/\sigma \to -\infty \\ t_0/\sigma^2 \to -\lambda}} R^{(3)}(t) = \lim \frac{\int_{(t-t_0)/\sigma}^{\infty} \exp\left(-\tfrac{1}{2}y^2\right) dy}{\int_{-t_0/\sigma}^{\infty} \exp\left(-\tfrac{1}{2}y^2\right) dy}$$

$$= \lim \frac{\exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right]}{\exp\left[-\frac{1}{2}\left(\frac{t_0}{\sigma}\right)^2\right]}$$

$$= \lim \exp\left[\frac{t_0 t}{\sigma^2}\left(1-\frac{1}{2}\frac{t}{t_0}\right)\right]$$

$$= \lim \exp\left(\frac{t_0 t}{\sigma^2}\right)$$

Thus,

$$\lim_{\substack{t_0/\sigma \to -\infty \\ t_0/\sigma^2 \to -\lambda}} R^{(3)}(t) = e^{-\lambda t} \tag{V.6}$$

For any value of t_0/σ, the truncated normal distribution satisfies the conditions previously required of a lifetime distribution, namely that $z(0)=f(0)>0$, and $z(t)$ is a monotonically increasing function of t. The latter is shown in Appendix C. The mean life associated with this distribution is obtained as follows:

$$t_m^{(3)} = \int_0^{\infty} t f^{(3)}(t) dt$$

$$= \frac{1}{P\left(\frac{-t_0}{\sigma}\right)} \frac{1}{\sigma\sqrt{2\pi}} \int_0^{\infty} t \exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right] dt$$

$$= t_0\left[1 + \frac{\exp\left(-\frac{1}{2}\frac{t_0^2}{\sigma^2}\right)}{\frac{t_0}{\sigma}\sqrt{2\pi}\; P\left(-\frac{t_0}{\sigma}\right)}\right] \tag{V.7}$$

or

$$t_m^{(3)} = t_0 \varphi^{(3)}(t_0/\sigma). \tag{V.8}$$

For the limiting cases discussed previously,

as $t_0/\sigma \to \infty$, $t_m^{(3)} \to t_0$,

as $t_0/\sigma \to -\infty$, such that $t_0/\sigma^2 \to -\lambda$, $t_m^{(3)} \to 1/\lambda$.

To normalize the distribution as a function of θ for any value of t_0/σ, we set $\theta_m = \theta_0 \varphi^{(3)}(\theta_0/\sigma') = 1$, and $\theta_0/\sigma' = t_0/\sigma$. Thus

$$\theta_0 = \frac{1}{\varphi^{(3)}(t_0/\sigma)}, \tag{V.9}$$

and

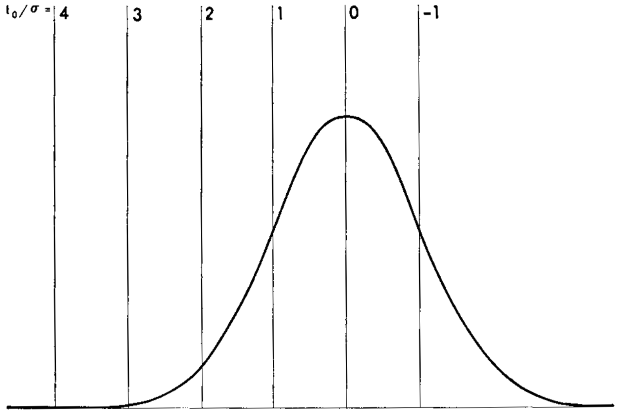


Figure 7 **Truncation of normal density function, showing position of $t=0$ corresponding to values of t_0/σ.**

$$\sigma' = \frac{1}{t_0/\sigma\,\varphi^{(3)}(t_0/\sigma)}\ . \tag{V.9}$$

Then

$$f^{*(3)}(\theta) = \frac{1}{P(-\theta_0/\sigma')} \cdot \frac{1}{\sigma'\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\theta-\theta_0}{\sigma'}\right)^2\right]\ . \tag{V.10}$$

The values of θ_0 and σ' for various values of t_0/σ are given in Table 2.

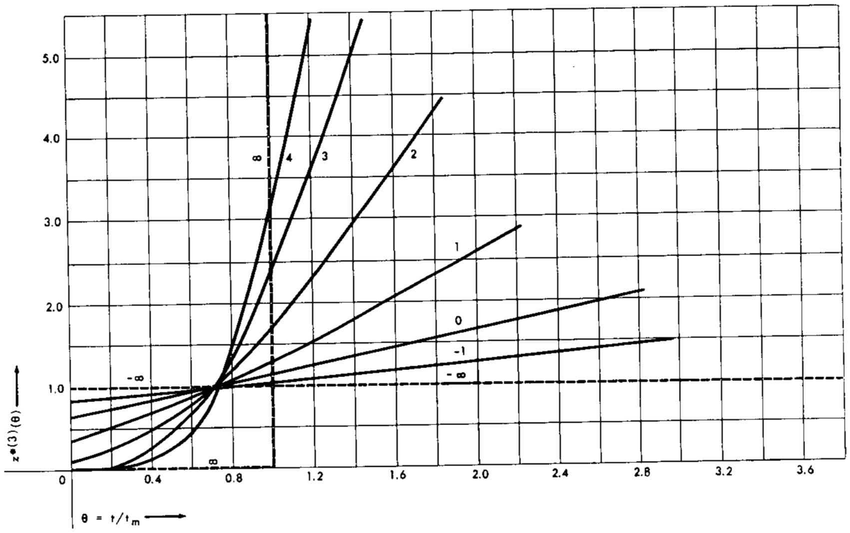
Table 2

	$F^{(3)}$		
$\theta_0/\sigma' = t_0/\sigma$	θ_0	σ'	H†
−1	−1.90	1.90	0.985
0	0	1.253	0.952
1	0.778	0.778	0.850
2	0.974	0.487	0.621
3	0.999	0.333	0.313
4	1.000	0.250	0.017

† This quantity is defined in Section VII.

In Figs. 8–10, $R^{*(3)}$, $z^{*(3)}$, and $f^{*(3)}$ are plotted as functions of θ for the above values of t_0/σ. The dotted lines represent the limiting cases of the step function and the exponential. A comparison of these curves with Figs. 1–6 reveals that this truncated normal distribution is capable of fitting a wider range of lifetime behavior than either of the other two-parameter distributions studied.

Figure 8
Hazard $z^{*(3)}(\theta)$
(truncated normal).
Values of θ_0/σ' indicated on curves.



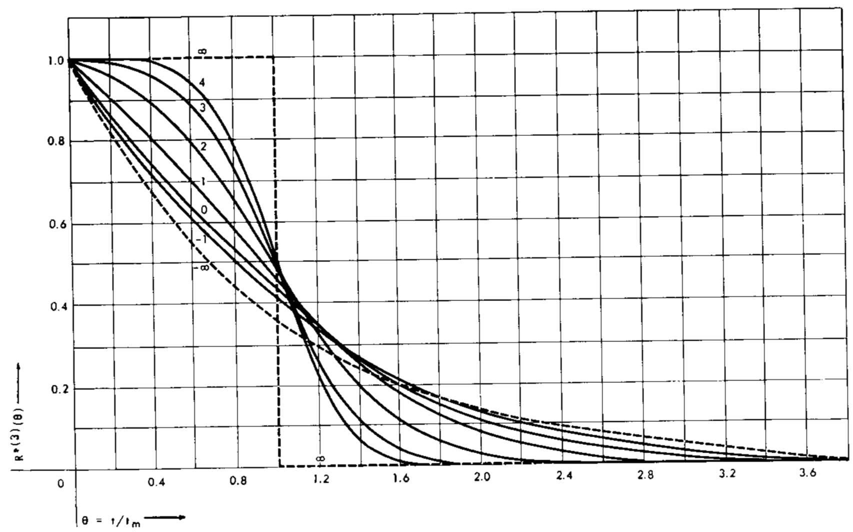
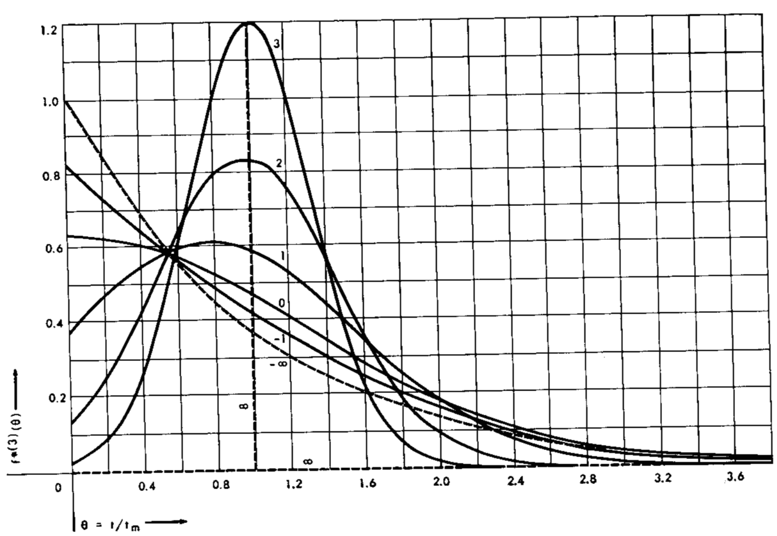


Figure 9
Survival probability $R^{*(3)}(\theta)$
(truncated normal).
Values of θ_0/σ' indicated on curves.

Figure 10
Lifetime density $f^{*(3)}(\theta)$
(truncated normal).
Values of θ_0/σ' indicated on curves.



VI. Numerical solution of the renewal equation for the proposed distributions

$U^*(\theta)$, the expected number of replacements for a component of age zero at $\theta=0$, has been numerically calculated as a function of normalized time $\theta=t/t_m$, for all the distributions discussed in the last two sections.

It was shown in Section II (Eqs. II.15 and II.16) that

$$U^*(\theta)=\sum_{i=1}^{\infty}U^*_i(\theta) \ , \qquad \text{(VI.1)}$$

where

$U^*_1(\theta)=F^*(\theta)$

and

$$U^*_i(\theta)=\int_0^\theta U^*_{i-1}(\theta-\tau)f^*(\tau)d\tau, \qquad i>1 \ .$$

To perform the numerical calculation, the last equation was approximated by the trapezoidal rule, giving:

$$U^*_i(\theta)=\frac{\Delta\tau}{2}\left[U^*_{i-1}(\theta)f(0)+U^*_{i-1}(0)f(\theta)\right]$$
$$+\sum_{j=1}^{n-1}U^*_{i-1}(\theta-\tau_j)f^*(\tau_j)\Delta\tau \ ,$$

where $\tau_j=j\Delta\tau$, $\tau_n=n\Delta\tau=\theta$. (VI.2)

The $U^*_i(\theta)$ were calculated with $\Delta\tau=0.2$ for θ ranging from 0 to 3. In Figs. 11–13, $U^*(\theta)$ is plotted for distributions (1), (2), and (3). It is estimated that these results are correct to within 1%. The relative rate of growth of the distribution functions $U^*_i(\theta)$ for successive renewals may be observed in Fig. 18, where U^*_2 is plotted against U^*_1 for the truncated normal distributions.

VII. Entropy as a measure of "randomness"

It was pointed out in Section III that the exponential law corresponds to maximum "randomness" of lifetimes. This arises from the fact that the age of an exponential-law component tells nothing about its subsequent probability of survival. However, we have adopted the more realistic assumption that $R(t;x)$ decreases with initial age x, i.e. hazard increases with time. This behavior is usually characterized as a wear-out effect. We have seen that we can obtain models ranging from an exponential law limit, corresponding to complete randomness, to a step function, corresponding to fixed lifetime or complete determinacy. It is clear that a measure of the degree of the wear-out effect or, what is the same thing, the deviation from exponential law behavior is needed.

In Statistical Mechanics[13] and Information Theory,[14] the quantity H, often called entropy, has been used as a measure of disorder or randomness. This quantity will serve as a measure of the "randomness" of lifetimes too. We define

$$H=-\int_0^\infty f^*(\theta)\ln f^*(\theta)d\theta \ , \qquad \text{(VII.1)}$$

where $f^*(\theta)$ is the lifetime density function, θ is time normalized so that the mean life is equal to 1. By the Calculus of Variations, it may be shown that, of all lifetime density functions with unit mean life, H is maximum for the exponential, $f^*(\theta)=e^{-\theta}$, and is then equal to 1.

For all the models considered in this paper, H decreases monotonically as the parameter ratios vary from their exponential-law limits to their "wear-out" limits. For the truncated normal distribution, as $t_0/\sigma\to\infty$ (the behavior approaches complete determinacy), H approaches $-\infty$.

The values of H as functions of the parameter ratios a/b, a/c, and t_0/σ are listed in Tables 1 and 2, and plotted in Figs. 19 and 20. $H^{(1)}$ and $H^{(2)}$ were obtained by numerical integration and $H^{(3)}$ was calculated from the result

$$H^{(3)}(\xi)=-\ln[\xi^2\eta(\eta-1)]+\tfrac{1}{2}(1-\xi^2\eta) \ , \qquad \text{(VII.2)}$$

where $\xi\equiv t_0/\sigma$,

$$\eta=1+\frac{\exp\left(-\frac{1}{2}\xi^2\right)}{\sqrt{2\pi}\,\xi P(-\xi)} \ .$$

The range of values of H gives a measure of the range of lifetime behavior which a model will describe. Thus, the three models discussed in this paper have limiting values of H displayed in Table 3.

Table 3

$F(t)$	*Max. H*	*Min. H*
$1-\exp[-(at+b^2t^2)]$	1 $(a/b=\infty)$	0.717 $(a/b=0)$
$1-\exp[-(at+c^3t^3)]$	1 $(a/c=\infty)$	0.399 $(a/b=0)$
Truncated normal	1 $(t_0/\sigma=-\infty)$	$-\infty$ $(t_0/\sigma=\infty)$

The three ranges of H give a numerical measure to our previous statement that the truncated normal distribution is capable of fitting a wider range of lifetime behavior than either of the other two-parameter models considered. It is interesting to note the range of values of H for other lifetime distributions, with increasing hazards, which are used as models in reliability theory. These are listed in Table 4.

Table 4

Distribution	*Entropy Range*
Weibull	$-\infty$ to 1
Gamma	$-\infty$ to 1
Log Normal	$-\infty$ to 0.919

Thus, as measured by "entropy," the Weibull and Gamma distributions describe as wide a range of lifetime behavior as does the truncated normal. However, as was pointed out in Section IV, these functions do not allow any flexibility in the choice of a value of $z(0)$.

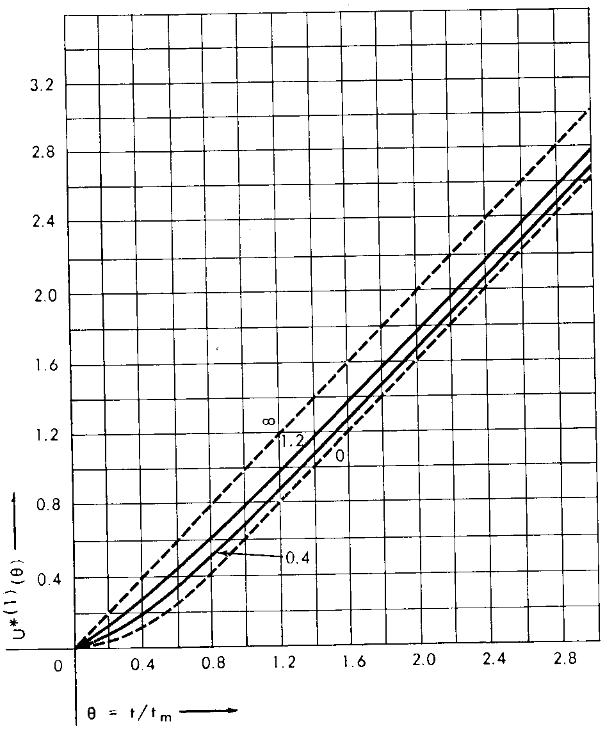


Figure 11

Expected number of replacements $U^{*(1)}(\theta)$.

Values of α/β indicated on curves.

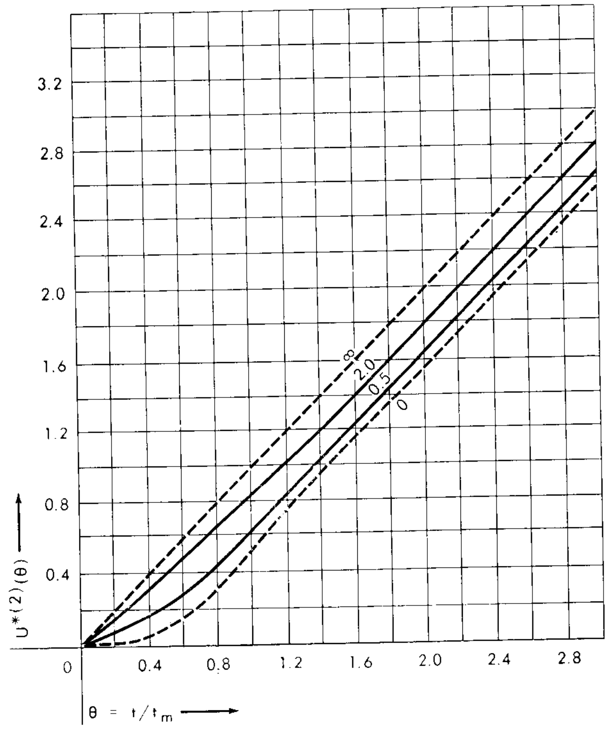


Figure 12

Expected number of replacements $U^{*(2)}(\theta)$.

Values of α/γ indicated on curves.

Figure 13

Expected number of replacements $U^{*(3)}(\theta)$.

Values of θ_0/σ' indicated on curves.

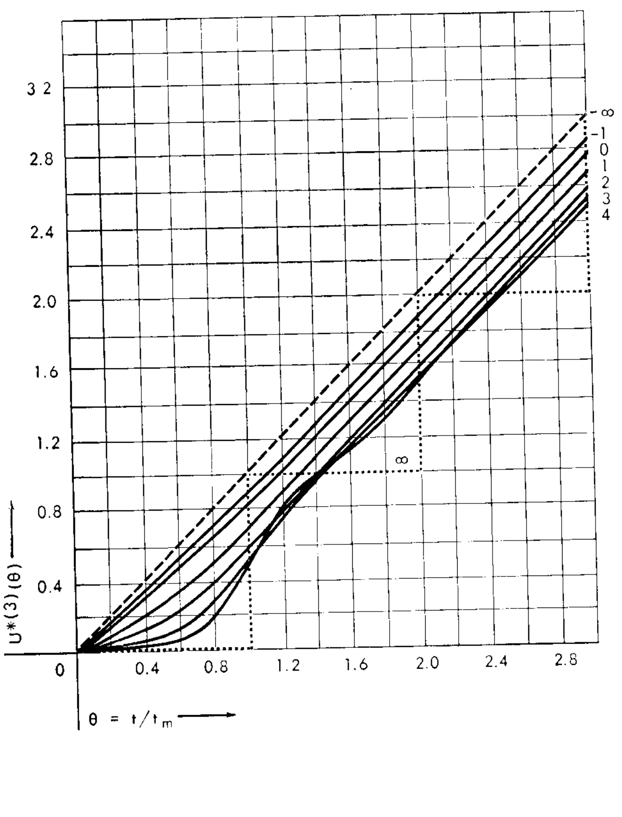
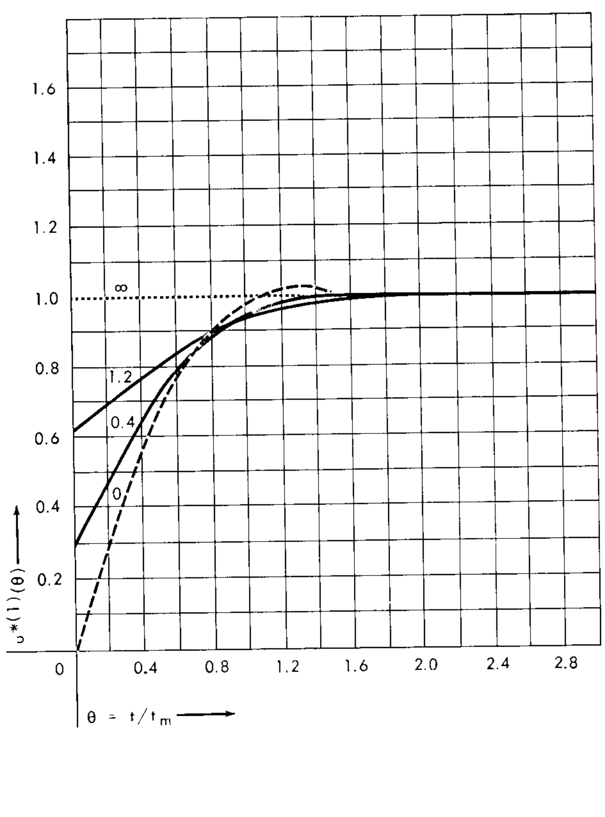


Figure 14

Renewal rate $u^{*(1)}(\theta)$.

Values of α/β indicated on curves.



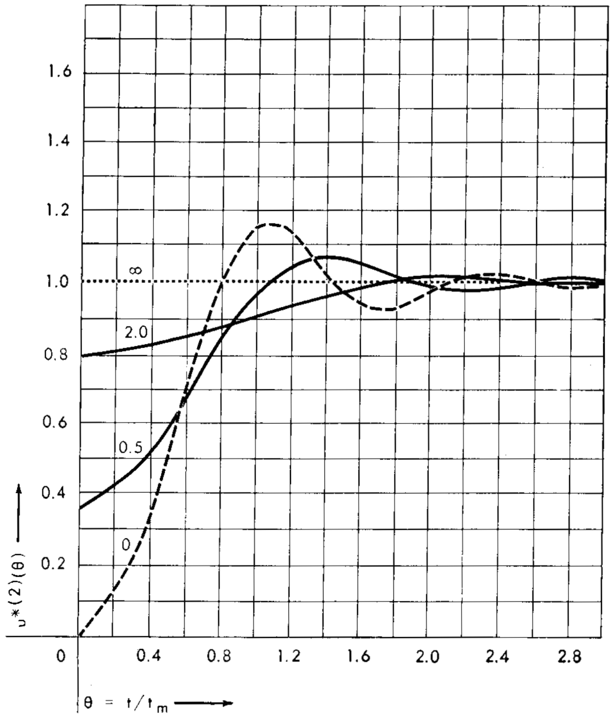


Figure 15

Renewal rate $u^{*(2)}(\theta)$.

Values of α/γ indicated on curves.

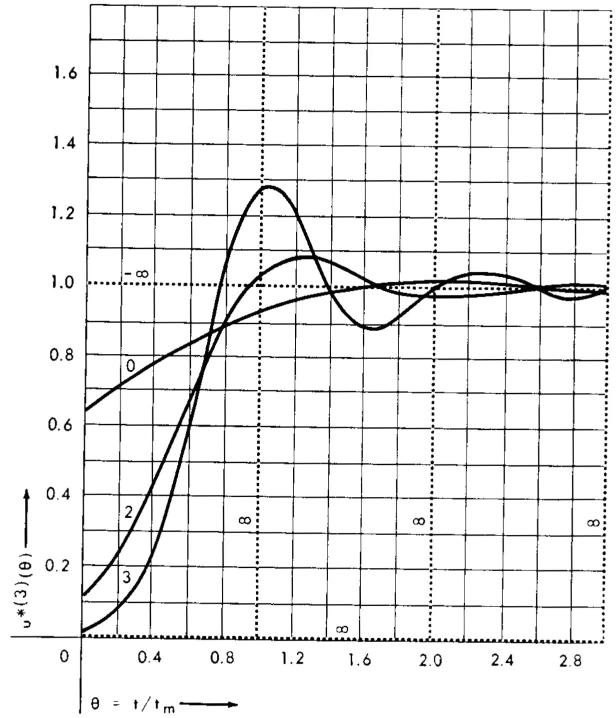


Figure 16

Renewal rate $u^{*(3)}(\theta)$.

Values of θ_0/σ' indicated on curves.

Figure 17

$U^{(3)}$ normalized to time over time at which $U^{(3)}(t) = 0.1$.

Values of t_0/σ indicated on curves.

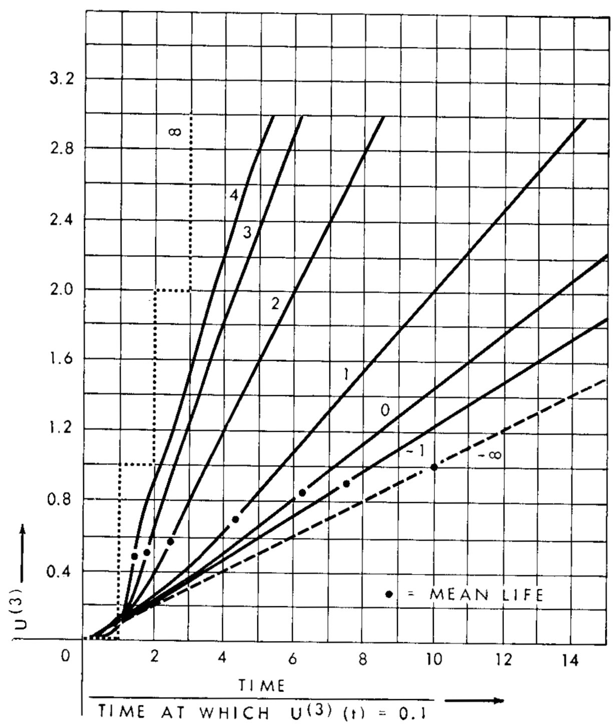
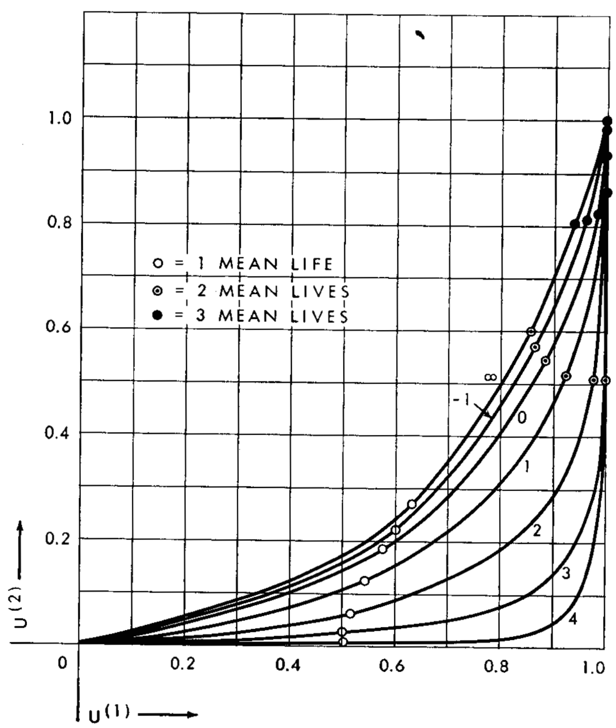


Figure 18

$U^{(2)}$ versus $U^{(1)}$.

Values of t_0/σ indicated on curves.



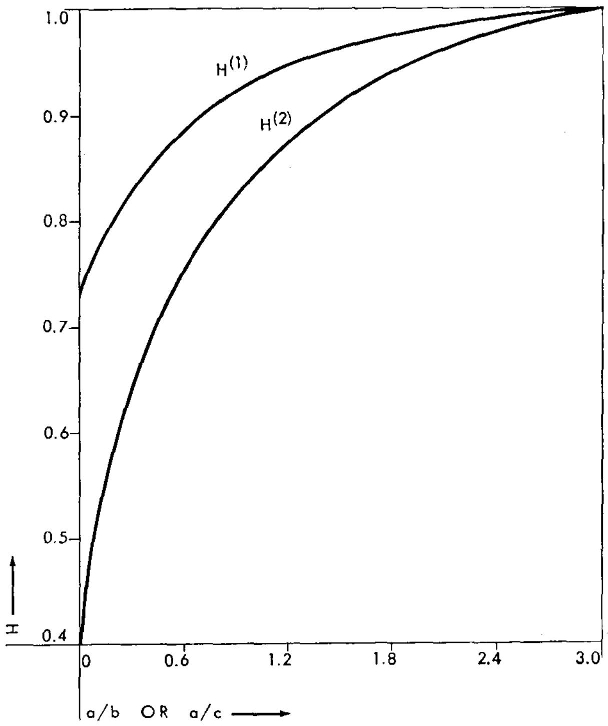


Figure 19
Entropy versus parameter ratios.
$H^{(1)}$ vs. a/b and $H^{(2)}$ vs. a/c.

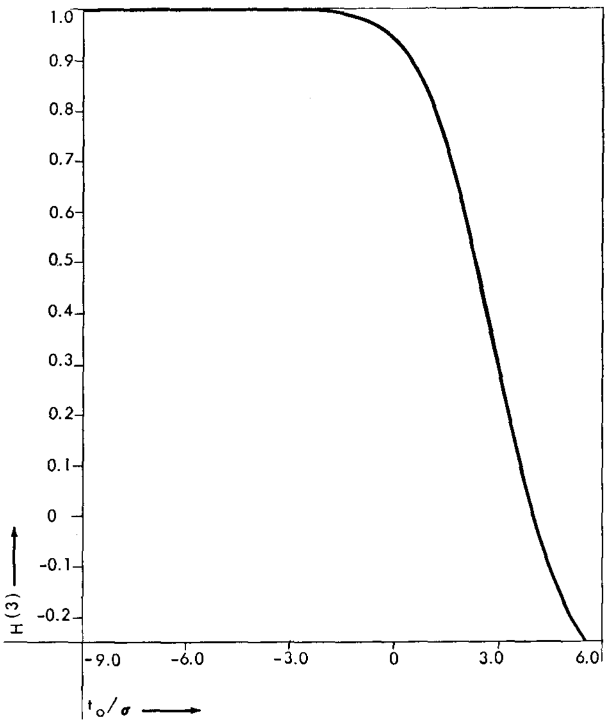


Figure 20
Entropy versus parameter ratio.
$H^{(3)}$ vs. t_0/σ.

VIII. Discussion of curves

On the accompanying graphs, probability of survival R^*, hazard z^*, lifetime density f^*, expected number of replacements U^*, and renewal rate u^*, have been plotted as functions of normalized time θ for the three models considered in this paper. In all cases, it was assumed that the initial age was zero. The plots have been made for selected values of the parameter ratios a/b, a/c, and t_0/σ, varying from the limits ($a/b=\infty$, $a/c=\infty$, $t_0/\sigma=-\infty$) corresponding to maximum entropy (exponential law), to the limits ($a/b=0$, $a/c=0$, $t_0/\sigma=\infty$) corresponding to the minimum entropy for the given models.

Figs. 1 and 2 illustrate the assumption of linear and quadratic increase of hazard with age. The dotted curves indicate the limits. Fig. 8 shows the hazard as a function of age for the truncated normal distribution. It is clear that the latter is capable of fitting a wider range of behavior than either of the other two.

In Figs. 3, 4, and 9, it may be observed that the survival probability for times short compared to the mean life increases as the parameter values depart from their exponential-law limits. Thus, for equal mean lives, those models with lower entropy have greater short-term survival probabilities than that associated with the exponential law.

Conversely, suppose available data provide information about survival probability over a short period of time, and a prediction of mean life is desired. The predicted value will vary widely with the value of the parameter ratio of the model used in prediction. For example, suppose it is known that the survival probability for the first 1000 hours of operation is 0.95. Then, an exponential distribution predicts a mean life of 19,600 hours. Any model with lower entropy predicts a shorter mean life. The truncated normal with $t_0/\sigma=4$ predicts a mean life of 1700 hours. *Thus, prediction of long-term behavior from short-term data without prior knowledge of the nature of the distribution is a meaningless process.*

In Figs. 11, 12, and 13, the numerically calculated values of $U^*(\theta)$, the expected number of replacements up to time $\theta=t/t_m$, are plotted. For the parameter values we have used, these curves clearly approximate straight lines of unit slope by the time $\theta=3$. However, in most cases of practical interest, the most significant portions of these curves correspond to values of θ less than one. For the exponential law, we have $U^*(\theta)=\theta$ for all $\theta\geq 0$. However, for any other model considered here, $U^*(\theta)<\theta$.

The approximate slopes of the $U^*(\theta)$ curves, representing $u^*(\theta)$, the renewal rates, are plotted in Figs. 14, 15, and 16. For exponential law, we have $u^*(\theta)=1$ for all $\theta\geq 0$. For all our other models, $u^*(0)=f^*(0)<1$, and for $\theta<1$, the renewal rate is a monotonically increasing function of θ. $u^*(\theta)$ reaches a peak value >1 for some value of $\theta>1$ and then oscillates with decreasing amplitude around its limiting value of 1. The amplitude of these oscillations increases as the values of the parameter ratios approach their "wear out" limits. Once again, these curves indicate that the truncated normal distribution is capable of representing the widest range of behavior.

When the expected number of replacements is plotted as a function of time normalized *in terms of the mean life of a component* (as in Fig. 13), the curves for various values of the parameter ratios are not startlingly different. However, it should be borne in mind that, in current practice, mean life is usually estimated from data obtained in a time that is a small fraction of the mean life. The fallacy of this type of prediction is illustrated in Fig. 17, in which $U^{(3)}$ is plotted as a function of time normalized in terms of the time at which $U(t)=0.1$.

Fig. 18 illustrates the relative behavior of successive generations in renewal processes governed by a truncated normal distribution with various values of t_0/σ. In each case, the distribution function for the second generation of components is plotted against the distribution for the first generation. It may be seen from the plot that, for an exponential distribution, when half the first generation has been replaced ($U_1=0.5$), more than 0.15 of the second generation has also been removed. As the entropy approaches $-\infty$, the value of U_2 corresponding to $U_1=0.5$ approaches zero. These curves are dependent only on the ratio t_0/σ and are independent of the time scale.

In conclusion, we propose that the truncated normal distribution be used as a model for fitting component removal data obtained in long-term reliability studies. This, of course, is based on our assumption that survival probability for a given time interval is a decreasing function of the initial age of a component. We feel that, under this assumption, this model provides the maximum flexibility for a two-parameter distribution.

Appendix A

Solution of the renewal rate equation

$$u(t;x)=f(t;x)+\int_0^t u(t-\tau;x)f(\tau)d\tau \ , \qquad \text{(II.18)}$$

by Laplace transforms :

Let $\bar{u}(s;x)=L\{u(t;x)\}$

$$=\int_0^\infty u(t;x)\exp(-st)dt, \text{ be the Laplace transform of } u(t;x).$$

Similarly $\bar{f}(s;x)\equiv L\{f(t;x)\}$.

Now, since

$$L\left\{\int_0^t u(t-\tau;x)f(\tau)\,d\tau\right\}=\bar{u}(s;x)\bar{f}(s) \ ,$$

we may transform both sides of the equation and obtain

$$\bar{u}(s;x)=\bar{f}(s;x)+\bar{f}(s)\bar{u}(s;x) \ , \qquad \text{or}$$

$$\bar{u}(s;x)=\frac{\bar{f}(s;x)}{1-\bar{f}(s)} \ .$$

This is the general Laplace transform solution of this type of integral equation. Now consider the special case of the exponential distribution, where

$$f(t;x)=f(t)=\lambda e^{-\lambda t} \ ,$$

$$\bar{f}(s;x)=\bar{f}(x)=L\left\{\lambda e^{-\lambda t}\right\}=\lambda/(s+\lambda) \ ,$$

and hence

$$\bar{u}(s;x)=\lambda/s \ .$$

Then, using the inverse Laplace transformation, we find

$$u(t;x)=u(t)=\lambda \ , \qquad t\geq 0 \ ;$$

$$U(t;x)=U(t)=\lambda t \ , \qquad t\geq 0 \ .$$

Appendix B

We will prove that for any given $t>0$, statement (1), $R(t;x)$ is a monotonically decreasing function of x, is equivalent to statement (2), $z(l)$ is a monotonically increasing function of l (the age of the component), by showing (*a*) that (1) implies (2) and conversely (*b*) that (2) implies (1).

a) Let

$$\frac{\partial R(t;x)}{\partial x}<0 \ , \qquad \text{all } t>0, \text{ all } x\geq 0 \ .$$

We have

$$\frac{\partial R(t;x)}{\partial x}=\frac{\partial}{\partial x}\,\frac{R(x+t)}{R(x)}=\frac{-R(x)f(x+t)+R(x+t)f(x)}{[R(x)]^2} \ ,$$

and hence

$$\frac{f(x+t)}{R(x+t)}>\frac{f(x)}{R(x)} \ .$$

That is,

$$z(x+t)>z(x) \ , \qquad \text{all } t>0, \text{ all } x\geq 0 \ ;$$

or

$$z(l_2)>z(l_1) \ , \qquad \text{all } l_2>l_1\geq 0 \ .$$

b) Let

$$z(l_2)>z(l_1) \ , \qquad \text{for any } l_2>l_1\geq 0 \ .$$

Then

$$\int_0^t z(x_2+\tau)d\tau>\int_0^t z(x_1+\tau)d\tau \ , \qquad \text{for } x_2>x_1\geq 0 \ .$$

That is,

$$\exp\left[-\int_{x_2}^{x_2+t} z(u)du\right] < \exp\left[-\int_{x_1}^{x_1+t} z(u)du\right]$$

or

$$R(t;x_2)<R(t;x_1) \ .$$

Appendix C

To prove that $z^{(3)}(t)$ is a monotonically increasing function of t:

By (V.3), we have

$$\frac{dz^{(3)}(t)}{dt}=\frac{\exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right]}{\left[\sigma\int_{(t-t_0)/\sigma}^{\infty}\exp\left(-\frac{1}{2}y^2\right)dy\right]^2}\times$$

$$\left\{\exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right]-\left(\frac{t-t_0}{\sigma}\right)\int_{(t-t_0)/\sigma}^{\infty}\exp\left(-\frac{1}{2}y^2\right)dy\right\}$$

The expression outside the curly brackets is positive for all t. The expression inside the cuily brackets equals

$$\int_{(t-t_0)/\sigma}^{\infty}\left[y-\left(\frac{t-t_0}{\sigma}\right)\right]\exp\left(-\frac{1}{2}y^2\right)dy\ .$$

Since the integrand is positive for all y greater than $(t-t_0)/\sigma$, this last integral must be positive. Therefore dz/dt is positive, and $z(t)$ is monotonically increasing.

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Received July 1958