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Irredundant Disjunctive and Conjunctive Forms of a Boolean Function

Abstract: A thorough algebraic method is described for the determination of the complete set of irredundant normal and conjunctive forms of a Boolean function. The method is mechanical and therefore highly programmable on a computer.

Introduction

Since C. E. Shannon's[1] work on the analysis and synthesis of relay and switching circuits, the simplification (minimization) of the Boolean expression symbolizing the operation of a binary system has become a major problem. W. V. Quine[2] has treated the problem of obtaining the simplest normal equivalent of a Boolean function. More generally, we define "Quine's problem" as the problem of finding the complete set of irredundant normal (or alternational or disjunctive) forms of a given function.

In the present paper, we describe a general method for the solution of Quine's problem. We also consider the problem of finding the complete set of irredundant conjunctive forms of a given function, and we show that this problem can be reduced to Quine's problem, through a set of transformations which we fully describe. Any solution to Quine's problem is therefore a solution to this problem, including the method we describe.

Definitions and symbols

Consider n binary independent variables. These are represented by the letters

$$v_{n-1}\, v_{n-2} \cdots v_i \cdots v_2\, v_1\, v_o.$$

The complement (inverse, or negation) of the variable v_i is written $\bar{v}_i$.

The symbol $+$ represents alternation (disjunction, inclusive union, logical sum, inclusive OR).

The symbol $\cdot$ represents conjunction (intersection, logical product, AND).

A binary function will be generally denoted by f.

f implies f^1 is written $f \rightarrow f^1$.

f is equivalent to f^1 is written $f = f^1$.

Single variables or inverses of single variables are referred to as *literals*.

It is always possible to replace a conjunction of literals where a given literal appears more than once by an equivalent conjunction where this literal appears only once, by virtue of the idempotent law. Such a conjunction will be referred to as conjunctional term, or simply *term*. (This is actually Quine's "fundamental formula.") Similarly, an alternation of literals where a given literal appears only once will be referred to as alternational term, or *alterm*.

A *normal* form (or alternational, or disjunctive form) is an alternation of terms.

A *conjunctive* form is a conjunction of alterms.

An *implicant* of a given function is a term which implies that function.

An *implicate* of a given function is an alterm which is implied by that function.

An implicant of a function of n variables where all n variables appear will be called *canonical implicant*.

An implicate of a function of n variables where all n variables appear will be called *canonical implicate*.

Following Quine, a term ϕ_1 is said to *subsume* a term ϕ_2 if and only if all the literals whereof ϕ_2 is a conjunction are among the literals whereof ϕ_1 is a conjunction. Similarly, an alterm ψ_1 is said to subsume an alterm ψ_2 if and only if all the literals whereof ψ_2 is an alternation are among the literals whereof ψ_1 is an alternation.

It is clear that if ϕ_1 subsumes ϕ_2, $\phi_1 \rightarrow \phi_2$, and that if ψ_1 subsumes ψ_2, $\psi_2 \rightarrow \psi_1$.

If ϕ_1, ϕ_2, ϕ_3 etc.... are implicants of a function f, the alternation $(\phi_1 + \phi_2 + \phi_3)$ is also an implicant of f. Similarly, if ψ_1, ψ_2, ψ_3 are implicates of a function f, the

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conjunction $(\psi_1\psi_2\psi_3 \ldots .)$ is also an implicate of f.

Also if $f^1 \rightarrow f$, we have $ff^1 = f^1$ and $f + f^1 = f$.

An irredundant alternation (or conjunction) is an alternation (or conjunction) of terms (or alterms) such that none of the terms (or alterms) involved is superfluous, and none of the literals within any of these terms (or alterms) is superfluous.

Quine defined a *prime implicant* of a function f as an implicant of f which subsumes no other implicant of f. We define a *prime implicate* of a function f as an implicate of f which subsumes no other implicate of f.

The set of prime implicants and the set of prime implicates of a given function f are finite. Quine has proved that any simplest normal equivalent of f will necessarily be an alternation of prime implicants of f.

This is easily extended to any irredundant normal form. In an identical manner, any irredundant conjunctive form will necessarily be a conjunction of prime implicates.

The set of irredundant normal forms

Given the complete set of implicants of a function f

$$\phi_1\ \phi_2 \ldots . \phi_j \ldots . \phi_n$$

Consider an irredundant solution

$$f_i = \phi_k + \phi_l + \phi_m$$

Any ϕ_j such that $j \neq k \neq l \neq m$ is superfluous with respect to f_i and satisfies the relation

$$\phi_j \rightarrow \phi_k + \phi_l + \phi_m \quad (j \neq k \neq l \neq m)$$

The problem is to determine the complete set of irredundant combinations of the prime implicants of the function f.

The set of irredundant conjunctive forms

Given the set of prime implicates of a function f

$$\psi_1\ \psi_2 \ldots . \psi_j \ldots . \psi_n$$

any irredundant conjunctive solution will necessarily be a conjunction of prime implicates of f.

The problem is to determine the complete set of irredundant combinations of these prime implicates of f.

Theorem

Given the complete set of prime implicates

$$\psi_1\ \psi_2 \ldots .. \psi_j \ldots .. \psi_n$$

of a function f, any irredundant conjunctive form f_i of f is the dual* of an irredundant normal form f_i^1 of an auxiliary function f^1 whose prime implicants are

$$\phi_1\ \phi_2 \ldots . \phi_j \ldots .. \phi_n$$

obtained from the set of prime implicates by the transformation $\phi_j = \overline{\psi}_j$

Proof

Consider an irredundant conjunctive form f_i given by

$$f_i = \psi_k\psi_l\psi_m$$

Regarding this solution, any ψ_j such that $j \neq k \neq l \neq m$ is superfluous, i.e.,

$$\psi_k\psi_l\psi_m = \psi_k\psi_l\psi_m \cdot \psi_j (j \neq k \neq l \neq m)$$

This gives

$$\psi_k\psi_l\psi_m \rightarrow \psi_j (j \neq k \neq l \neq m)$$

i.e., any prime implicate implied by the conjunction f_i is superfluous with respect to this conjunction.

Again, this gives

$$\overline{\psi}_j \rightarrow \overline{\psi}_k + \overline{\psi}_l + \overline{\psi}_m (j \neq k \neq l \neq m)$$

Introducing the functions

$$\phi_1\ \phi_2 \ldots .. \phi_j \ldots .. \phi_n$$

such that $\phi_j = \overline{\psi}_j$

we get

$$\phi_j \rightarrow \phi_k + \phi_l + \phi_m (j \neq k \neq l \neq m)$$

Now, since every ψ_j is an alterm, it follows that every ϕ_j is a term. Again, since no alterm ψ_j subsumes another, no term ϕ_j will subsume another. We can therefore consider the set of ϕ_j as being the set of prime implicants of an auxiliary function which we shall denote f^1.

Now, since $\phi_j \rightarrow \phi_k + \phi_l + \phi_m$, one irredundant normal form of f^1 will be

$$f_i^1 = \phi_k + \phi_l + \phi_m$$

and we have $f_i = \bar{f}^1{}_i$

In other words, *given* the set of prime implicates ψ_j, first obtain the set ϕ_j such that $\phi_i = \overline{\psi}_j$. Then consider this set as being the set of prime implicants of a function f^1. Obtain the set of irredundant normal forms f_i^1 of this function f^1. The set of irredundant conjunctive forms will be derived from it by the simple transformation $f_i = \bar{f}_i^1$, i.e., replacing each ϕ in the solution f_i by the corresponding ψ, and alternation signs by conjunction signs.

The problem of finding the complete set of irredundant conjunctive forms is therefore reduced to Quine's problem, by the use of the above transformations. Any method providing a solution to Quine's problem, includ-

*Every literal is replaced by its inverse, every conjunctive sign by an alternation sign, and vice versa.

ing the method we are about to describe, is therefore valid in this case.

Remark

The set of prime implicates of f can be obtained by simply applying the distributive law

$$ab + cd = (a + c)(a + d)(b + c)(b + d)$$

to the alternation of the complete set of prime implicants or to some normal form of f, and suppressing any alterm subsuming another alterm.

Conversely, the set of prime implicants of f can be obtained by simply applying the distributive law

$$(a + b)(c + d) = ac + ad + bc + bd$$

to the conjunction of the complete set of prime implicates or to some conjunctive form of f, and suppressing any term subsuming another term.

The method

Let us first introduce the following fundamental operation: We define the *ratio* of a function f to a literal l as the binary function which is equivalent to f whenever l equals 1. The ratio of f to l is written

$$\frac{f}{l} \tag{1}$$

Examples

$$\frac{v_1v_2\bar{v}_3}{v_2} = v_1\bar{v}_3 \qquad \frac{v_1\bar{v}_3}{v_2} = v_1\bar{v}_3$$

$$\frac{v_1v_2\bar{v}_3}{v_3} = 0 \qquad \frac{v_1}{v_1} = 1 \qquad \frac{v_1}{v_1v_2} = 1 \qquad \text{etc.}$$

In other words, the ratio of f to l is obtained by forming the conjunction of f and l, and suppressing l from this conjunction.

It is easily shown that

$$\frac{f_1}{l_1} + \frac{f_2}{l_1} = \frac{f_1 + f_2}{l_1} \tag{2}$$

It follows that the ratio of a function f to a term which implies f is valid.

Proof

Let this term be ϕ. We have $\phi \rightarrow f$

therefore $\quad \phi f = \phi$

and $\quad \dfrac{f}{\phi} = \dfrac{f}{\phi f} = 1$

Consider now the list of prime implicants of a function f

$$\phi_1\,\phi_2 \cdots\cdots \phi_i\,\phi_j \cdots\cdots \phi_n$$

According to Quine, a given prime implicant is dispensable if it implies the alternation of the remaining prime implicants. In other words, a given prime implicant ϕ_i is dispensable if the alternation of the ratios ϕ_j/ϕ_i for all values of j except $j = i$, is valid.

Construct a chart as follows:

$$\begin{array}{c|ccccccc}
 & \phi_1 & \phi_2 & \cdots & \cdots & \phi_j & \cdots & \phi_n \\
\hline
\phi_1 & \bullet & \frac{\phi_2}{\phi_1} & \cdots & \cdots & \frac{\phi_j}{\phi_1} & \cdots & \frac{\phi_n}{\phi_1} \\
\phi_2 & \frac{\phi_1}{\phi_2} & \bullet & \cdots & \cdots & \frac{\phi_{j-}}{\phi_2} & \cdots & \frac{\phi_n}{\phi_2} \\
\cdots & \cdots & \cdots & \bullet & \cdots & \cdots & \cdots & \cdots \\
\phi_i & \frac{\phi_1}{\phi_i} & \frac{\phi_2}{\phi_i} & \cdots & \bullet & \frac{\phi_j}{\phi_i} & \cdots & \frac{\phi_n}{\phi_i} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \bullet & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \bullet & \cdots \\
\phi_n & \frac{\phi_1}{\phi_n} & \frac{\phi_2}{\phi_n} & \cdots & \cdots & \frac{\phi_j}{\phi_n} & \cdots & \bullet
\end{array}$$

The general term, on Row i and Column j is ϕ_j/ϕ_i. The diagonal is the locus of $\phi_i/\phi_i = 1$. Nothing should be written upon it.

A given prime implicant ϕ_i is dispensable if the alternation of all the ratios on Row i is valid. Let us consider an example:

The prime implicants of a function f are the following:

$$\bar{x}y \qquad x\bar{y} \qquad xz \qquad yz \qquad wz$$

The ratio chart is as follows:

		1 $\bar{x}y$	2 $x\bar{y}$	3 xz	4 yz	5 wz	
1	$\bar{x}y$	·	·	·	z	wz	Not dispensable
2	$x\bar{y}$	·	·	z	·	wz	Not dispensable
3	xz	·	$\bar{y}$	·	y	w	Dispensable
4	yz	$\bar{x}$	·	x	·	w	Dispensable
5	wz	$\bar{x}y$	$x\bar{y}$	x	y	·	Not dispensable

The prime implicants ϕ_1, ϕ_2 and ϕ_5 are not dispensable.
ϕ_3 is dispensable since $\bar{y} + y + w = 1$.
ϕ_4 is dispensable since $\bar{x} + x + w = 1$.

Consider now ϕ_3. From the chart, we can see that the alternation $\bar{y} + y + w$ is valid, which gives

$$\phi_3 \rightarrow \phi_2 + \phi_4 + \phi_5 \tag{3}$$

But also the alternation $\bar{y} + y$ is valid, which gives

$$\phi_3 \rightarrow \phi_2 + \phi_4 \tag{4}$$

This shows that ϕ_5 is not necessary to make ϕ_3 dispensable. Relation 3 is therefore redundant in ϕ_5 and Relation 4 is irredundant.

Similarly, Row 5 gives us the irredundant relation

$$\phi_4 \rightarrow \phi_1 + \phi_3 \tag{5}$$

Let us now introduce what we refer to as *presence factor* and denote σ_i. A presence factor σ_i is a binary coefficient, attached to ϕ_i such that the presence of prime implicant ϕ_i in a given irredundant normal form of f corresponds to $\sigma_i = 1$, and its absence to $\sigma_i = 0$.

Considering the relation $\phi_3 \rightarrow \phi_2 + \phi_4$, it follows that the *absence* of prime implicant ϕ_3 from a given irredundant normal form of f *implies* the presence of *both* ϕ_2 and ϕ_4 in that same normal form. In logical symbols, this gives

$$\overline{\sigma}_3 \rightarrow \sigma_2 \cdot \sigma_4$$

Note that the redundant relation $\phi_3 \rightarrow \phi_2 + \phi_4 + \phi_5$ would yield the relation $\overline{\sigma}_3 \rightarrow \sigma_2 \cdot \sigma_4 \cdot \sigma_5$ which is not irredundant, since the absence of ϕ_3 does not *necessarily* imply the presence of ϕ_5.

In a similar way, we can write the set of irredundant relations:

$$\overline{\sigma}_1 \rightarrow 0$$

$$\overline{\sigma}_2 \rightarrow 0$$

$$\overline{\sigma}_3 \rightarrow \sigma_2\sigma_4$$

$$\overline{\sigma}_4 \rightarrow \sigma_1\sigma_3$$

$$\overline{\sigma}_5 \rightarrow 0$$

We know, from the properties of implication that if $a \rightarrow b$, then $\bar{a}+b = 1$. This gives the set of equivalences:

$$\sigma_1 = 1$$

$$\sigma_2 = 1$$

$$\sigma_3 + \sigma_2\sigma_4 = 1$$

$$\sigma_4 + \sigma_1\sigma_3 = 1$$

$$\sigma_5 = 1$$

Any irredundant normal form of f has to satisfy *all* the above equivalences. In other words, we have an irredundant normal form every time the above equivalences are *simultaneously* satisfied. So that if we write:

$$S = \sigma_1 \cdot \sigma_2 \cdot (\sigma_3+\sigma_2\sigma_4) \cdot (\sigma_4+\sigma_1\sigma_3) \cdot \sigma_5 \tag{6}$$

We have an irredundant normal form of f whenever $S = 1$. S is called *presence function* of f. Computing (6), we get

$$S = \sigma_1\sigma_2\sigma_3\sigma_5 + \sigma_1\sigma_2\sigma_4\sigma_5 \tag{7}$$

The two irredundant normal forms of f are therefore

$$F_1 = \phi_1 + \phi_2 + \phi_3 + \phi_5 \qquad \text{for } \sigma_1\sigma_2\sigma_3\sigma_5 = 1$$

$$\text{and } F_2 = \phi_1 + \phi_2 + \phi_4 + \phi_5 \qquad \text{for } \sigma_1\sigma_2\sigma_4\sigma_5 = 1$$

These are

$$F_1 = \bar{x}y + x\bar{y} + xz + wz$$

$$\text{and } F_2 = \bar{x}y + x\bar{y} + yz + wz$$

In general, if the number of prime implicants of a function f is n, the chart yields n equivalences of the type $\overline{\sigma}_i \rightarrow S_i$, that is,

$$\sigma_i + S_i = 1 \tag{8}$$

Where S_i is a function of the presence factors σ_j for all values of j except $j = i$. The presence function S is therefore of the form

$$S = \prod_{i=1}^{n} (\sigma_i + S_i) \tag{9}$$

It involves no inverses of σ_i and therefore if the repeated conjunction is computed throughout, the only simplification which is necessary consists in suppressing any resulting term which subsumes another term.

An essential condition is that S_i be irredundant. In the above example, these S_i were rather obvious. It can happen, however, that the situation be more complicated, as the following example shows:

Consider the following chart of ϕ_j/ϕ_i. (Such a chart will be called ϕ chart.)

	1 $\bar{d}e$	2 $cd\bar{e}$	3 $\bar{a}cd$	4 $\bar{a}ce$	5 $a\bar{b}d$	6 $a\bar{b}e$	7 $\bar{b}cd$	8 $\bar{b}ce$
1 $\bar{d}e$	·	·	·	$\bar{a}c$	·	$a\bar{b}$	·	$\bar{b}c$
2 $cd\bar{e}$	·		$\bar{a}$	·	$a\bar{b}$	·	$\bar{b}$	·
3 $\bar{a}cd$	·	$\bar{e}$	·	e	·	·	$\bar{b}$	$\bar{b}e$
4 $\bar{a}ce$	$\bar{d}$	·	d	·	·	·	$\bar{b}d$	$\bar{b}$
5 $a\bar{b}d$	·	$c\bar{e}$	·	·	·	e	c	ce
6 $a\bar{b}e$	$\bar{d}$	·	·	·	d	·	cd	c
7 $\bar{b}cd$	·	$\bar{e}$	$\bar{a}$	$\bar{a}e$	a	ae	·	e
8 $\bar{b}ce$	$\bar{d}$	·	$\bar{a}d$	$\bar{a}$	ad	a	d	·

The first row yields $\quad \overline{\sigma}_1 \rightarrow 0 \quad S_1 = 0 \quad \sigma_1 = 1$

The second yields $\quad \overline{\sigma}_2 \rightarrow 0 \quad S_2 = 0 \quad \sigma_2 = 1$

The third yields $\quad \overline{\sigma}_3 \rightarrow \sigma_2\sigma_4 \quad S_3 = \sigma_2\sigma_4 \quad \sigma_3 + \sigma_2\sigma_4 = 1$

The fourth yields $\bar{\sigma}_4 \rightarrow \sigma_1\sigma_3 \quad S_3 = \sigma_1\sigma_3 \quad \sigma_4 + \sigma_1\sigma_3 = 1$

The fifth yields $\bar{\sigma}_5 \rightarrow 0 \quad S_5 = 0 \quad \sigma_5 = 1$

The sixth yields $\bar{\sigma}_6 \rightarrow \sigma_1\sigma_5 \quad S_6 = \sigma_1\sigma_5 \quad \sigma_6 + \sigma_1\sigma_5 = 1$

Now S_7 and S_8 are not obvious. To determine them, we resort to a procedure which we call *cracking*. First consider Row 7. Construct a ratio chart whose horizontal and vertical entries are the ratios on Row 7. (Such a chart will be denoted ϕ/ϕ_7 chart.)

	1	2	3	4	5	6	7	8
ϕ/ϕ_7	·	$\bar{e}$	$\bar{a}$	$\bar{a}e$	a	ae	·	e
1 ·	·	·	·	·	·	·	·	·
2 $\bar{e}$	·	·	$\bar{a}$	·	a	·	·	·
3 $\bar{a}$	·	$\bar{e}$	·	e	·	·	·	e
4 $\bar{a}e$	·	·	1	·	·	·	·	1
5 a	·	$\bar{e}$	·	·	·	e	·	e
6 ae	·	·	·	·	1	·	·	1
7 ·	·	·	·	·	·	·	·	·
8 e	·	·	$\bar{a}$	$\bar{a}$	a	a	·	·

Now, we can say that ϕ_7 implies the function considered, whenever the alternation of one of the lines in the above chart is valid. This gives us

$$\bar{\sigma}_7 \rightarrow (\sigma_2 + \sigma_3\sigma_5)(\sigma_3 + \sigma_2\sigma_4 + \sigma_2\sigma_8)(\sigma_4 + \sigma_3 + \sigma_8)$$
$$(\sigma_5 + \sigma_2\sigma_6 + \sigma_2\sigma_8)(\sigma_6 + \sigma_5 + \sigma_8)$$
$$(\sigma_8 + \sigma_3\sigma_5 + \sigma_3\sigma_6 + \sigma_4\sigma_5 + \sigma_4\sigma_6)$$

that is,

$$\bar{\sigma}_7 \rightarrow \sigma_3\sigma_5 + \sigma_2\sigma_8 + \sigma_2\sigma_3\sigma_6 + \sigma_2\sigma_4\sigma_5 + \sigma_2\sigma_4\sigma_6$$

Similarly for Row 8, if we construct the ϕ/ϕ_8 chart, we get:

$$\bar{\sigma}_8 \rightarrow \sigma_1\sigma_7 + \sigma_4\sigma_6 + \sigma_1\sigma_3\sigma_5 + \sigma_1\sigma_4\sigma_5 + \sigma_1\sigma_3\sigma_6$$

Finally, we can write an expression for S:

$$S = (\sigma_1)(\sigma_2)(\sigma_3 + \sigma_2\sigma_4)(\sigma_4 + \sigma_1\sigma_3)(\sigma_5)(\sigma_6 + \sigma_1\sigma_5)$$
$$(\sigma_7 + \sigma_3\sigma_5 + \sigma_2\sigma_8 + \sigma_2\sigma_3\sigma_6 + \sigma_2\sigma_4\sigma_5 + \sigma_2\sigma_4\sigma_6)$$
$$(\sigma_8 + \sigma_1\sigma_7 + \sigma_4\sigma_6 + \sigma_1\sigma_3\sigma_5 + \sigma_1\sigma_4\sigma_5 + \sigma_1\sigma_3\sigma_6)$$

Computing, we get

$$S = \sigma_1\sigma_2\sigma_3\sigma_5 + \sigma_1\sigma_2\sigma_4\sigma_5$$

The two irredundant solutions are thus:

$$F_1 = \bar{d}e + cd\bar{e} + \bar{a}cd + a\bar{b}d \qquad \text{and}$$

$$F_2 = \bar{d}e + cd\bar{e} + \bar{a}ce + a\bar{b}d$$

In other words, a presence function S is given by (9) where i is any row of the ϕ chart, and S_i a corresponding function of $\sigma_1, \sigma_2 \cdots \sigma_j \cdots \sigma_n (j \neq i)$.

In turn, S_i is given by

$$S_i = \prod_{k=1}^{n} (\sigma_k + S_{ik}) \tag{10}$$

where k is any row on the ϕ/ϕ_i chart, and S_{ik} a corresponding function of $\sigma_1, \sigma_2 \ldots . \sigma_j \ldots .. \sigma_n (j \neq i \neq k)$.

Similarly S_{ik} can be given by

$$S_{ik} = \prod_{l=1}^{n} (\sigma_l + S_{ikl}) \tag{11}$$

where l is any row on the $(\phi/\phi_i)/(\phi_k/\phi_i)$ chart and S_{ikl} a corresponding function of $\sigma_1, \sigma_2 \ldots . \sigma_j \ldots . \sigma_n$

$(j \neq i \neq k \neq l)$ ... etc.

If this method is used by a computer, cracking can be achieved in an exhaustive manner. But, for pencil and paper work, cracking is seldom needed, and the reader can check that even when the number of variables is very high, it is easy to obtain any function S_i. For instance, Row 7 of the ϕ chart shows

that $\phi_7 \rightarrow \phi_3 + \phi_5$ because $\bar{a} + a = 1$

also $\phi_7 \rightarrow \phi_2 + \phi_8$ because $\bar{e} + e = 1$

also $\phi_7 \rightarrow \phi_2 + \phi_3 + \phi_6$ because $\bar{e} + \bar{a} + ae = 1$

also $\phi_7 \rightarrow \phi_2 + \phi_4 + \phi_5$ because $\bar{e} + \bar{a}e + a = 1$

also $\phi_7 \rightarrow \phi_2 + \phi_4 + \phi_6$ because $\bar{e} + \bar{a}e + ae = 1$

This gives

$$\bar{\sigma}_7 \rightarrow \sigma_3\sigma_5 + \sigma_2\sigma_8 + \sigma_2\sigma_3\sigma_6 + \sigma_2\sigma_4\sigma_5 + \sigma_2\sigma_4\sigma_6$$

Conclusion

The present method is thorough and yields a complete list of all the *irredundant* forms of a given function. Of these, the simplest will be retained if desired, according to whatever criterion of simplicity is chosen. The procedure itself involves no choice and can therefore be entirely mechanized.

The main idea in this method is that whenever it is stated that a prime implicant is dispensable, this statement is, of course, true but often involves redundancy. To be irredundant the statement has to involve the complete list of reasons which are necessary and sufficient to make this prime implicant dispensable. Any reason implying another reason would appear in the S function as a term subsuming another term and would therefore be rejected. When such a list is not obvious, systematic "cracking" is used, if the problem is actually cracked into a number of minor problems of standard type.

This method can therefore be highly mechanized. However, if the solution is to be manual, it is considerably simpler than other existing ones, because the only difficulty would lie in writing the function S with a minimum of cracking, and this difficulty is actually of a very limited character.

Interesting papers on this general subject have been published by various authors, and especially R. H. Urbano, E. W. Samson, R. K. Mueller[3,4,5] and S. R. Petrick,[6] of the Air Research and Development Command. Petrick's paper presents a method for the direct determination of the irredundant forms of a Boolean function from the set of prime implicants, in which it is necessary to obtain the set of canonical implicants. In the present method however, only the set of prime implicants is necessary.

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