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Constructive Logics. Part I: A Tutorial on Proof Systems and Typed λ-Calculi

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May 1991

Publication Notes

This work was done while the author was on sabbatical leave from the University of Pennsylvania at Digital PRL.

Abstract

The purpose of this paper is to give an exposition of material dealing with constructive logics, typed λ-calculi, and linear logic. The emergence in the past ten years of a coherent field of research often named "logic and computation" has had two major (and related) effects: firstly, it has rocked vigorously the world of mathematical logic; secondly, it has created a new computer science discipline, which spans from what is traditionally called theory of computation, to programming language design. Remarkably, this new body of work relies heavily on some "old" concepts found in mathematical logic, like natural deduction, sequent calculus, and λ-calculus (but often viewed in a different light), and also on some newer concepts. Thus, it may be quite a challenge to become initiated to this new body of work (but the situation is improving, there are now some excellent texts on this subject matter). This paper attempts to provide a coherent and hopefully "gentle" initiation to this new body of work. We have attempted to cover the basic material on natural deduction, sequent calculus, and typed λ-calculus, but also to provide an introduction to Girard's linear logic, one of the most exciting developments in logic these past five years. The first part of these notes gives an exposition of background material (with the exception of the Girard-translation of classical logic into intuitionistic logic, which is new). The second part is devoted to linear logic and proof nets.

Résumé

Le but de cet article est de donner une présentation d'éléments de logique constructive, de lambda calcul typé, et de logique linéaire. L'émergence, ces dix dernières années, d'un domaine cohérent de recherche souvent appelé "logique et calcul" a eu deux effets majeurs (et concommitents): tout d'abord, elle a dynamisé le monde de la logique mathématique; deuxièmement, elle a créée une nouvelle discipline d'informatique, discipline qui s'étend depuis ce qu'on appelle traditionellement la théorie de la calculabilité à la conception des langages de programmation. Remarquablement, ce corps de connaissances repose en grande partie sur certains "vieux" concepts de logique mathématique, tel que la déduction naturelle, le calcul des séquents, et le λ-calcul (mais souvent vus avec une optique différente), et d'autres concepts plus nouveaux. Il est donc assez difficile de s'initier à ce nouveau domaine de recherche (mais la situation s'est améliorée depuis l'apparition d'excellents livres sur ce sujet). Cet article essaye de présenter "en douceur" et de façon cohérente ce corps de travaux. Nous avons essayé de couvrir des sujets classiques tels que la déduction naturelle, le calcul des séquents, et le λ-calcul typé, mais aussi de donner une introduction à la logique linéaire de Girard, un des dévelopements en logique les plus interessants de ces cinq dernières années. Dans une première partie nous présentons les bases (à l'exception de la traduction de Girard de la logique classique en logique intuitionniste, qui est nouvelle). La logique linéaire et les réseaux de preuves sont traités dans la deuxième partie.

Keywords

Natural deduction, lambda calculus, sequent calculus, linear logic.

Acknowledgements

I wish to thank Hassan Aït-Kaci, Andreas Podelski, and Ascánder Súarez, for their comments. Special thanks to Kathleen Milsted, Marcin Skubiszewski, and Jean-Christophe Patat, for proofreading earlier versions very carefully.

Contents

1 Introduction

The purpose of this paper is to give an exposition of material dealing with constructive logics, typed λ-calculi, and linear logic. During the last fifteen years, a significant amount of research in the areas of programming language theory, automated deduction, and more generally logic and computation, has relied heavily on concepts and results found in the fields of constructive logics and typed λ-calculi. However, there are very few comprehensive and introductory presentations of constructive logics and typed λ-calculi for noninitiated researchers, and many people find it quite frustrating to become acquainted to this type of research. Our motivation in writing this paper is to help fill this gap. We have attempted to cover the basic material on natural deduction, sequent calculus, and typed λ-calculus, but also to provide an introduction to Girard's linear logic [7], one of the most exciting developments in logic these past five years. As a consequence, we discovered that the amount of background material necessary for a good understanding of linear logic was quite extensive, and we found it convenient to break this paper into two parts. The first part gives an exposition of background material (with the exception of the Girard-translation of classical logic into intuitionistic logic, which is new [9]). The second part is devoted to linear logic and proof nets.

In our presentation of background material, we have tried to motivate the introduction of various concepts by showing that they are indispensable to achieve certain natural goals. For pedagogical reasons, it seems that it is best to begin with proof systems in natural deduction style (originally due to Gentzen [3] and thoroughly investigated by Prawitz [14] in the sixties). This way, it is fairly natural to introduce the distinction between intuitionistic and classical logic. By adopting a description of natural deduction in terms of judgements, as opposed to the tagged trees used by Gentzen and Prawitz, we are also led quite naturally to the encoding of proofs as certain typed λ-terms, and to the correspondence between proof normalization and β-conversion (the *Curry/Howard isomorphism* [10]). Sequent calculi can be motivated by the desire to obtain more "symmetric" systems, but also systems in which proof search is easier to perform (due to the subformula property). At first, the cut rule is totally unnecessary and even undesirable, since we are trying to design systems as deterministic as possible. We then show how every proof in the sequent calculus ($\mathcal{G}_i$) can be converted into a natural deduction proof (in $\mathcal{N}_i$). In order to provide a transformation in the other direction, we introduce the cut rule. But then, we observe that there is a mismatch, since we have a transformation $\mathcal{N}: \mathcal{G}_i \to \mathcal{N}_i$ on cut-free proofs, whereas $\mathcal{G}: \mathcal{N}_i \to \mathcal{G}_i^{cut}$ maps to proofs possibly with cuts. The mismatch is resolved by Gentzen's fundamental *cut elimination theorem*, which in turn singles out the crucial role played by the *contraction rule*. Indeed, the contraction rule plays a crucial role in the proof of the cut elimination theorem, and furthermore it cannot be dispensed with in intuitionistic logic (with some exceptions, as shown by some recent work of Lincoln, Scedrov, and Shankar [12]). We are thus setting the stage for linear logic, in which contraction (and weakening) are dealt with in a very subtle way. We then investigate a number of sequent calculi that allow us to prove the decidability of provability in propositional classical logic and in propositional intuitionistic logic. The cut elimination theorem is proved in full for the Gentzen system $\mathcal{LK}$ using Tait's induction measure [18], and some twists due to Girard [8]. We conclude with a fairly extensive discussion of the reduction of classical logic to intuitionistic

logic. Besides the standard translations due to Gödel, Gentzen, and Kolmogorov, we present an improved translation due to Girard [9] (based on the notion of polarity of a formula).

2 Natural Deduction and Simply-Typed λ-Calculus

We first consider a syntactic variant of the natural deduction system for implicational propositions due to Gentzen [3] and Prawitz [14].

In the natural deduction system of Gentzen and Prawitz, a deduction consists in deriving a proposition from a finite number of packets of assumptions, using some predefined inference rules. Technically, packets are multisets of propositions. During the course of a deduction, certain packets of assumptions can be "closed", or "discharged". A proof is a deduction such that all the assumptions have been discharged. In order to formalize the concept of a deduction, one faces the problem of describing rigorously the process of discharging packets of assumptions. The difficulty is that one is allowed to discharge any number of occurrences of the same proposition in a single step, and this requires some form of tagging mechanism. At least two forms of tagging techniques have been used.

- The first one, used by Gentzen and Prawitz, consists in viewing a deduction as a tree whose nodes are labeled with propositions. One is allowed to tag any set of occurrences of some proposition with a natural number, which also tags the inference that triggers the simultaneous discharge of all the occurrences tagged by that number.

- The second solution consists in keeping a record of all undischarged assumptions at every stage of the deduction. Thus, a deduction is a tree whose nodes are labeled with expressions of the form $\Gamma \vdash A$, called *sequents*, where A is a proposition, and Γ is a record of all undischarged assumptions at the stage of the deduction associated with this node.

Although the first solution is perhaps more natural from a human's point of view and more economical, the second one is mathematically easier to handle. In the sequel, we adopt the second solution. It is convenient to tag packets of assumptions with labels, in order to discharge the propositions in these packets in a single step. We use variables for the labels, and a packet consisting of occurrences of the proposition A is written as $x\colon A$. Thus, in a sequent $\Gamma \vdash A$, the expression Γ is any finite set of the form $x_1\colon A_1, \ldots, x_m\colon A_m$, where the x_i are pairwise distinct (but the A_i need not be distinct). Given $\Gamma = x_1\colon A_1, \ldots, x_m\colon A_m$, the notation $\Gamma, x\colon A$ is only well defined when $x \neq x_i$ for all i, $1 \leq i \leq m$, in which case it denotes the set $x_1\colon A_1, \ldots, x_m\colon A_m, x\colon A$. We have the following axioms and inference rules.

Definition 1 *The axioms and inference rules of the system $\mathcal{N}_m^{\supset}$ (minimal implicational logic) are listed below:*

$$\Gamma, x\colon A \vdash A$$

$$\frac{\Gamma, x\colon A \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\text{-intro})$$

$$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \quad (\supset\textit{-elim})$$

In an application of the rule ($\supset$*-intro*), we say that the proposition A which appears as a hypothesis of the deduction is *discharged* (or *closed*). It is important to note that the ability to label packets consisting of occurrences of the same proposition with different labels is essential, in order to be able to have control over which groups of packets of assumptions are discharged simultaneously. Equivalently, we could avoid tagging packets of assumptions with variables if we assumed that in a sequent $\Gamma \vdash C$, the expression Γ, also called a *context*, is a *multiset* of propositions. The following two examples illustrate this point.

Example 2.1 Let

$$\Gamma = x\colon A \supset (B \supset C), y\colon A \supset B, z\colon A.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash A \supset (B \supset C) \quad \Gamma \vdash A}{\Gamma \vdash B \supset C} \quad \dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}}{x\colon A \supset (B \supset C), y\colon A \supset B, z\colon A \vdash C}}{x\colon A \supset (B \supset C), y\colon A \supset B \vdash A \supset C}}{x\colon A \supset (B \supset C) \vdash (A \supset B) \supset (A \supset C)}}{\vdash \big(A \supset (B \supset C)\big) \supset \big((A \supset B) \supset (A \supset C)\big)}$$

In the above example, two occurrences of A are discharged simultaneously. Compare with the example below where these occurrences are discharged in two separate steps.

Example 2.2 Let

$$\Gamma = x\colon A \supset (B \supset C), y\colon A \supset B, z_1\colon A, z_2\colon A.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash A \supset (B \supset C) \quad \Gamma \vdash A}{\Gamma \vdash B \supset C} \quad \dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}}{x\colon A \supset (B \supset C), y\colon A \supset B, z_1\colon A, z_2\colon A \vdash C}}{x\colon A \supset (B \supset C), y\colon A \supset B, z_1\colon A \vdash A \supset C}}{x\colon A \supset (B \supset C), z_1\colon A \vdash (A \supset B) \supset (A \supset C)}}{z_1\colon A \vdash \big(A \supset (B \supset C)\big) \supset \big((A \supset B) \supset (A \supset C)\big)}}{\vdash A \supset \Big(\big(A \supset (B \supset C)\big) \supset \big((A \supset B) \supset (A \supset C)\big)\Big)}$$

For the sake of comparison, we show what these two natural deductions look like in the system of Gentzen and Prawitz, where packets of assumptions discharged in the same inference are tagged with a natural number. Example 2.1 corresponds to the following tree:

Example 2.3

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A \supset (B \supset C))^3 \qquad A^1}{B \supset C} \qquad \dfrac{(A \supset B)^2 \qquad A^1}{B}}{C}\,1}{A \supset C}\,2}{(A \supset B) \supset (A \supset C)}\,3}{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}$$

and Example 2.2 to the following tree:

Example 2.4

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A \supset (B \supset C))^3 \qquad A^1}{B \supset C} \qquad \dfrac{(A \supset B)^2 \qquad A^4}{B}}{C}\,1}{A \supset C}\,2}{(A \supset B) \supset (A \supset C)}\,3}{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}\,4$$

$$A \supset \Big((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))\Big)$$

It is clear that a context (the Γ in a sequent $\Gamma \vdash A$) is used to tag packets of assumptions and to record the time at which they are discharged. From now on, we stick to the presentation of natural deduction using sequents.

Proofs may contain redundancies, for example when an elimination immediately follows an introduction, as in the following example:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, x\colon A \vdash B\end{array}}{\Gamma \vdash A \supset B} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash A\end{array}}{\Gamma \vdash B}$$

Intuitively, it should be possible to construct a deduction for $\Gamma \vdash B$ from the two deductions $\mathcal{D}_1$ and $\mathcal{D}_2$ without using at all the hypothesis $x\colon A$. This is indeed the case. If we look closely at the deduction $\mathcal{D}_1$, from the shape of the inference rules, assumptions are never created, and the leaves must be labeled with expressions of the form $\Gamma, \Delta, x\colon A, y\colon C \vdash C$ or $\Gamma, \Delta, x\colon A \vdash A$, where $y \neq x$. We can form a new deduction for $\Gamma \vdash B$ as follows: in $\mathcal{D}_1$, wherever a leaf of the form $\Gamma, \Delta, x\colon A \vdash A$ occurs, replace it by the deduction obtained from $\mathcal{D}_2$ by adding Δ to the premise of each sequent in $\mathcal{D}_2$. Actually, one should be careful to first make a fresh copy of $\mathcal{D}_2$ by renaming all the variables so that clashes with variables in $\mathcal{D}_1$ are avoided. Finally, delete the assumption $x\colon A$ from the premise of every sequent in the resulting proof. The resulting deduction is obtained by a kind of substitution and may be denoted as $\mathcal{D}_1[\mathcal{D}_2/x]$, with some minor abuse of notation. Note that the assumptions $x\colon A$ occurring in the leaves of the form $\Gamma, \Delta, x\colon A, y\colon C \vdash C$ were never used anyway. This illustrates the fact that not all assumptions are necessarily used. This will not be the case in linear logic [7]. Also, the same assumption may be used more than once, as we can see in the ($\supset$-*elim*) rule. Again, this will not be the case in linear logic, where every assumption is used exactly once, unless specified otherwise by an explicit mechanism. The step which consists in transforming the above redundant proof figure into the deduction $\mathcal{D}_1[\mathcal{D}_2/x]$ is called a *reduction step* or *normalization step*.

We now show that the simply-typed λ-calculus provides a natural notation for proofs in natural deduction, and that β-conversion corresponds naturally to proof normalization. The trick is to annotate inference rules with terms corresponding to the deductions being built, by placing these terms on the righthand side of the sequent, so that the conclusion of a sequent appears to be the "type of its proof". This way, inference rules have a reading as "type-checking rules". This discovery due to Curry and Howard is known as the *Curry/Howard isomorphism*, or *formulae-as-types principle* [10]. Furthermore, and this is the deepest aspect of the Curry/Howard isomorphism, proof normalization corresponds to term reduction in the λ-calculus associated with the proof system.

Definition 2 *The type-checking rules of the λ-calculus $\lambda^{\supset}$ (simply-typed λ-calculus) are listed below:*

$$\Gamma, x\colon A \vdash x\colon A$$

$$\dfrac{\Gamma, x\colon A \vdash M\colon B}{\Gamma \vdash (\lambda x\colon A.\, M)\colon A \supset B} \quad \textit{(abstraction)}$$

$$\dfrac{\Gamma \vdash M\colon A \supset B \quad \Gamma \vdash N\colon A}{\Gamma \vdash (MN)\colon B} \quad \textit{(application)}$$

Now, sequents are of the form $\Gamma \vdash M : A$, where M is a simply-typed λ-term representing a deduction of A from the assumptions in Γ. Such sequents are also called *judgements*, and Γ is called a *type assignment* or *context*.

The example of redundancy is now written as follows:

$$\dfrac{\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\ M) : A \supset B} \qquad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x : A.\ M)N : B}$$

Now, $\mathcal{D}_1$ is incorporated in the deduction as the term M, and $\mathcal{D}_2$ is incorporated in the deduction as the term N. The great bonus of this representation is that $\mathcal{D}_1[\mathcal{D}_2/x]$ corresponds to $M[N/x]$, the result of performing a β-reduction step on $(\lambda x : A.\ M)N$.

Thus, the simply-typed λ-calculus arises as a natural way to encode natural deduction proofs, and β-reduction corresponds to proof normalization. The correspondence between proof normalization and term reduction is the deepest and most fruitful aspect of the Curry/Howard isomorphism. Indeed, using this correspondence, results about the simply-typed λ-calculus can be translated in terms of natural deduction proofs, a very nice property.

When we deal with the calculus $\lambda^{\supset}$, rather than using $\supset$, we usually use $\rightarrow$, and thus, the calculus is denoted as $\lambda^{\rightarrow}$. In order to avoid ambiguities, the delimiter used to separate the lefthand side from the righthand side of a judgement $\Gamma \vdash M : A$ will be $\triangleright$, so that judgements are written as $\Gamma \triangleright M : A$.

3 Adding Conjunction, Negation, and Disjunction

First, we present the natural deduction systems, and then the corresponding extensions of the simply-typed λ-calculus. As far as proof normalization is concerned, conjunction does not cause any problem, but as we will see, negation and disjunction are more problematic. In order to add negation, we add the new constant $\bot$ (**false**) to the language, and define negation $\neg A$ as an abbreviation for $A \supset \bot$.

Definition 3 *The axioms and inference rules of the system $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ (intuitionistic propositional logic) are listed below:*

$$\Gamma, x : A \vdash A$$

$$\dfrac{\Gamma, x : A \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\text{-intro})$$

$$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \quad (\supset\text{-elim})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad (\wedge\textit{-intro})$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \quad (\wedge\textit{-elim}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \quad (\wedge\textit{-elim})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad (\vee\textit{-intro}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \quad (\vee\textit{-intro})$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, x\colon A \vdash C \quad \Gamma, y\colon B \vdash C}{\Gamma \vdash C} \quad (\vee\textit{-elim})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \quad (\bot\textit{-elim})$$

Minimal propositional logic $\mathcal{N}_m^{\supset,\wedge,\vee,\bot}$ is obtained by dropping the ($\bot$-*elim*) rule. In order to obtain the system of *classical propositional logic*, denoted $\mathcal{N}_c^{\supset,\wedge,\vee,\bot}$, we add to $\mathcal{N}_m^{\supset,\wedge,\vee,\bot}$ the following inference rule corresponding to the principle of proof by contradiction (*by-contra*) (also called *reductio ad absurdum*).

$$\frac{\Gamma, x\colon \neg A \vdash \bot}{\Gamma \vdash A} \quad (\textit{by-contra})$$

Several useful remarks should be made.

(1) In classical propositional logic ($\mathcal{N}_c^{\supset,\wedge,\vee,\bot}$), the rule

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \quad (\bot\textit{-elim})$$

can be derived, since if we have a deduction of $\Gamma \vdash \bot$, then for any arbitrary A we have a deduction $x\colon \neg A, \Gamma \vdash \bot$, and thus a deduction of $\Gamma \vdash A$ by applying the (*by-contra*) rule.

(2) The proposition $A \supset \neg\neg A$ is derivable in $\mathcal{N}_m^{\supset,\wedge,\vee,\bot}$, but the reverse implication $\neg\neg A \supset A$ is not derivable, even in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$. On the other hand, $\neg\neg A \supset A$ is derivable in $\mathcal{N}_c^{\supset,\wedge,\vee,\bot}$:

$$\dfrac{\dfrac{\dfrac{x\colon \neg\neg A, y\colon \neg A \vdash \neg\neg A \qquad x\colon \neg\neg A, y\colon \neg A \vdash \neg A}{x\colon \neg\neg A, y\colon \neg A \vdash \bot} \quad (\textit{by-contra})}{x\colon \neg\neg A \vdash A}}{\vdash \neg\neg A \supset A}$$

(3) Using the (*by-contra*) inference rule together with ($\supset$-*elim*) and ($\vee$-*intro*), we can prove $\neg A \vee A$ (that is, $(A \supset \bot) \vee A$). Let

$$\Gamma = x\colon ((A \supset \bot) \vee A) \supset \bot,\ y\colon A.$$

We have the following proof for $(A \supset \bot) \vee A$.

$$\dfrac{\dfrac{x\colon ((A \supset \bot) \vee A) \supset \bot \vdash ((A \supset \bot) \vee A) \supset \bot \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash A}{\Gamma \vdash ((A \supset \bot) \vee A) \supset \bot \quad \Gamma \vdash (A \supset \bot) \vee A}}{x\colon ((A \supset \bot) \vee A) \supset \bot, y\colon A \vdash \bot}}{x\colon ((A \supset \bot) \vee A) \supset \bot \vdash A \supset \bot}}{x\colon ((A \supset \bot) \vee A) \supset \bot \vdash (A \supset \bot) \vee A}}{\dfrac{x\colon ((A \supset \bot) \vee A) \supset \bot \vdash \bot}{\vdash (A \supset \bot) \vee A}}\ (\textit{by-contra})$$

The typed λ-calculus $\lambda^{\rightarrow,\times,+,\bot}$ corresponding to $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ is given in the following definition.

Definition 4 *The typed λ-calculus $\lambda^{\rightarrow,\times,+,\bot}$ is defined by the following rules.*

$$\Gamma, x\colon A \triangleright x\colon A$$

$$\dfrac{\Gamma, x\colon A \triangleright M\colon B}{\Gamma \triangleright (\lambda x\colon A.\ M)\colon A \to B}\ \textit{(abstraction)}$$

$$\dfrac{\Gamma \triangleright M\colon A \to B \quad \Gamma \triangleright N\colon A}{\Gamma \triangleright (M\,N)\colon B}\ \textit{(application)}$$

$$\dfrac{\Gamma \triangleright M\colon A \quad \Gamma \triangleright N\colon B}{\Gamma \triangleright \langle M, N\rangle\colon A \times B}\ \textit{(pairing)}$$

$$\dfrac{\Gamma \triangleright M\colon A \times B}{\Gamma \triangleright \pi_1(M)\colon A}\ \textit{(projection)} \qquad \dfrac{\Gamma \triangleright M\colon A \times B}{\Gamma \triangleright \pi_2(M)\colon B}\ \textit{(projection)}$$

$$\dfrac{\Gamma \triangleright M\colon A}{\Gamma \triangleright \mathtt{inl}(M)\colon A + B}\ \textit{(injection)} \qquad \dfrac{\Gamma \triangleright M\colon B}{\Gamma \triangleright \mathtt{inr}(M)\colon A + B}\ \textit{(injection)}$$

$$\dfrac{\Gamma \triangleright P\colon A + B \quad \Gamma, x\colon A \triangleright M\colon C \quad \Gamma, y\colon B \triangleright N\colon C}{\Gamma \triangleright \mathtt{case}(P, \lambda x\colon A.\ M, \lambda y\colon B.\ N)\colon C}\ \textit{(by-cases)}$$

$$\dfrac{\Gamma \triangleright M\colon \bot}{\Gamma \triangleright \triangle_A(M)\colon A}\ (\bot\textit{-elim})$$

A syntactic variant of $\mathtt{case}(P, \lambda x\colon A.\ M, \lambda y\colon B.\ N)$ often found in the litterature is $\mathtt{case}\ P\ \mathtt{of}\ \mathtt{inl}(x\colon A) \Rightarrow M \mid \mathtt{inr}(y\colon B) \Rightarrow N$, or even $\mathtt{case}\ P\ \mathtt{of}\ \mathtt{inl}(x) \Rightarrow M \mid \mathtt{inr}(y) \Rightarrow N$, and the (*by-cases*) rule can be written as

$$\frac{\Gamma \triangleright P\colon A + B \quad \Gamma, x\colon A \triangleright M\colon C \quad \Gamma, y\colon B \triangleright N\colon C}{\Gamma \triangleright (\mathtt{case}\ P\ \mathtt{of}\ \mathtt{inl}(x\colon A) \Rightarrow M \mid \mathtt{inr}(y\colon B) \Rightarrow N)\colon C} \quad (\textit{by-cases})$$

We also have the following reduction rules.

Definition 5 *The reduction rules of the system $\lambda^{\rightarrow,\times,+,\perp}$ are listed below:*

$$\begin{aligned}
(\lambda x\colon A.\ M)N &\longrightarrow M[N/x],\\
\pi_1(\langle M, N\rangle) &\longrightarrow M,\\
\pi_2(\langle M, N\rangle) &\longrightarrow N,\\
\mathtt{case}(\mathtt{inl}(P), \lambda x\colon A.\ M, \lambda y\colon B.\ N) &\longrightarrow M[P/x], \quad \textit{or}\\
\mathtt{case}\ \mathtt{inl}(P)\ \mathtt{of}\ \mathtt{inl}(x\colon A) \Rightarrow M \mid \mathtt{inr}(y\colon B) \Rightarrow N &\longrightarrow M[P/x],\\
\mathtt{case}(\mathtt{inr}(P), \lambda x\colon A.\ M, \lambda y\colon B.\ N) &\longrightarrow N[P/y], \quad \textit{or}\\
\mathtt{case}\ \mathtt{inr}(P)\ \mathtt{of}\ \mathtt{inl}(x\colon A) \Rightarrow M \mid \mathtt{inr}(y\colon B) \Rightarrow N &\longrightarrow N[P/y],\\
\triangle_{A\to B}(M)N &\longrightarrow \triangle_B(M),\\
\pi_1(\triangle_{A\times B}(M)) &\longrightarrow \triangle_A(M),\\
\pi_2(\triangle_{A\times B}(M)) &\longrightarrow \triangle_B(M),\\
\mathtt{case}(\triangle_{A+B}(P), \lambda x\colon A.\ M, \lambda y\colon B.\ N) &\longrightarrow \triangle_C(P),\\
\triangle_A(\triangle_\perp(M)) &\longrightarrow \triangle_A(M).
\end{aligned}$$

Alternatively, as suggested by Ascánder Súarez, we could replace the rules for $\mathtt{case}$ by the rules

$$\begin{aligned}
\mathtt{case}(\mathtt{inl}(P), M, N) &\longrightarrow MP,\\
\mathtt{case}(\mathtt{inr}(P), M, N) &\longrightarrow NP,\\
\mathtt{case}(\triangle_{A+B}(P), M, N) &\longrightarrow \triangle_C(P).
\end{aligned}$$

A fundamental result about natural deduction is the fact that every proof (term) reduces to a normal form, which is unique up to α-renaming. This result was first proved by Prawitz [15] for the system $\mathcal{N}_i^{\supset,\wedge,\vee,\perp}$.

Theorem 1 (Church-Rosser property, Prawitz (1971)) *Reduction in $\lambda^{\rightarrow,\times,+,\perp}$ (specified in Definition 5) is confluent. Equivalently, conversion in $\lambda^{\rightarrow,\times,+,\perp}$ is Church-Rosser.*

A proof can be given by adapting the method of Tait and Martin-Löf [13] using a form of parallel reduction (see also Stenlund [16]).

Theorem 2 (Strong normalization property, Prawitz (1971)) *Reduction in $\lambda^{\rightarrow,\times,+,\bot}$ (as in Definition 5) is strongly normalizing.*

A proof can be given by adapting Tait's reducibility method [17], [19], as done in Girard [5] (1971), [6] (1972) (see also Gallier [2]).

If one looks at the rules of the system $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ (or $\lambda^{\rightarrow,\times,+,\bot}$), one notices a number of unpleasant features:

(1) There is an *asymmetry* between the lefthand side and the righthand side of a sequent (or judgement): the righthand side must consist of a single formula, but the lefthand side may have any finite number of assumptions. This is typical of intuitionistic logic, but it is also a defect.

(2) Negation is very badly handled, only in an indirect fashion.

(3) The ($\supset$-*intro*) rule and the ($\vee$-*elim*) rule are global rules requiring the discharge of assumptions.

(4) Worse of all, the ($\vee$-*elim*) rule contains the parasitic formula C which has nothing to do with the disjunction being eliminated.

Finally, note that it is quite difficult to search for proofs in such a system. Gentzen's sequent systems remedy some of these problems.

4 Gentzen's Sequent Calculi

The main idea is that now, a sequent $\Gamma \vdash \Delta$ consists of two finite multisets Γ and Δ of formulae, and that rather than having introduction and elimination rules, we have rules introducing a connective on the left or on the right of a sequent. A first version of such a system for classical propositional logic is given next. In these rules Γ and Δ stand for possibly empty finite multisets of propositions.

Definition 6 *The axioms and inference rules of the system $\mathcal{G}_c^{\supset,\wedge,\vee,\neg}$ for classical propositional logic are given below.*

$$A, \Gamma \vdash \Delta, A$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \textit{(contrac: left)} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \quad \textit{(contrac: right)}$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad (\wedge\textit{: left)} \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \quad (\wedge\textit{: right)}$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \ (\vee\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \ (\vee\text{: } \textit{right})$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \ (\supset\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B} \ (\supset\text{: } \textit{right})$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \ (\neg\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \ (\neg\text{: } \textit{right})$$

Note the perfect symmetry of the left and right rules. If one wants to deal with the extended language containing also $\bot$, one needs to add the axiom

$$\bot, \Gamma \vdash \Delta.$$

One might be puzzled and even concerned about the presence of the contraction rule. Indeed, one might wonder whether the presence of this rule will not cause provability to be undecidable. This would certainly be quite bad, since we are only dealing with propositions! Fortunately, it can be shown that the contraction rule is redundant for classical propositional logic. But then, why include it in the first place? The main reason is that it cannot be dispensed with in intuitionistic logic, or in the case of quantified formulae. (Recent results of Lincoln, Scedrov, and Shankar [12], show that in the case of propositional intuitionistic restricted to implications, it is possible to formulate a contraction-free system which easily yields the decidability of provability). Since we would like to view intuitionistic logic as a subsystem of classical logic, we cannot eliminate the contraction rule from the presentation of classical systems. Another important reason is that the contraction rule plays an important role in cut elimination. Although it is possible to hide it by dealing with sequents viewed as pairs of sets rather than multisets, we prefer to deal with it explicitly. Finally, the contraction rule plays a crucial role in linear logic, and in the understanding of the correspondence between proofs and computations, in particular strict versus lazy evaluation.

In order to obtain a system for intuitionistic logic, we restrict the righthand side of a sequent to consist of *at most one formula*. We also modify the ($\supset$: *left*) rule and the ($\vee$: *right*) rule which splits into two rules. The (*contrac*: *right*) rule disappears, and it is also necessary to add a rule of weakening on the right, to mimic the ($\bot$-*elim*) rule.

Definition 7 *The axioms and inference rules of the system $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$ for intuitionistic propositional logic are given below.*

$$A, \Gamma \vdash A$$

$$\frac{\Gamma \vdash}{\Gamma \vdash A} \ (\textit{weakening}\text{: } \textit{right})$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \ (\textit{contrac}\text{: } \textit{left})$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \ (\wedge\text{: } \textit{left}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \ (\wedge\text{: } \textit{right})$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \ (\vee\text{: } \textit{left})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \ (\vee\text{: } \textit{right}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \ (\vee\text{: } \textit{right})$$

$$\frac{\Gamma \vdash A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \ (\supset\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \ (\supset\text{: } \textit{right})$$

$$\frac{\Gamma \vdash A}{\neg A, \Gamma \vdash} \ (\neg\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \ (\neg\text{: } \textit{right})$$

In the above rules, Δ contains at most one formula. If one wants to deal with the extended language containing also $\bot$, one simply needs to add the axiom

$$\bot, \Gamma \vdash \Delta,$$

where Δ contains at most one formula. If we choose the language restricted to formulae over $\wedge$, $\supset$, $\vee$, and $\bot$, then negation $\neg A$ is viewed as an abbreviation for $A \supset \bot$. Such a system can be simplified a little bit if we observe that the axiom $\bot, \Gamma \vdash \Delta$ implies that the rule

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$

is derivable. Indeed, assume that we have the axiom $\bot, \Gamma \vdash \Delta$. If $\Gamma \vdash \bot$ is provable, since no inference rule applies to $\bot$, the leaf nodes of this proof must be of the form $\Gamma' \vdash \bot$. Thus, we must have $\bot \in \Gamma'$, in which case $\Gamma' \vdash A$ is an axiom. Thus, we obtain a proof of $\Gamma \vdash A$. We can also prove that the converse almost holds. Since $\bot, \Gamma \vdash \bot$ is an axiom, using the rule

$$\frac{\bot, \Gamma \vdash \bot}{\bot, \Gamma \vdash A}$$

we see that $\bot, \Gamma \vdash A$ is provable. The reason why this is not exactly the converse is that $\bot, \Gamma \vdash$ is not provable in this system. This suggests to consider sequents of the form $\Gamma \vdash A$ where A consists *exactly of a single formula*. In this case, the axiom $\bot, \Gamma \vdash A$ is equivalent to the rule

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \ (\bot\text{: } \textit{right})$$

We have the following system.

Definition 8 *The axioms and inference rules of the system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ for intuitionistic propositional logic are given below.*

$$A, \Gamma \vdash A$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \quad (\bot\text{: } \textit{right})$$

$$\frac{A, A, \Gamma \vdash C}{A, \Gamma \vdash C} \quad (\textit{contrac}\text{: } \textit{left})$$

$$\frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C} \quad (\wedge\text{: } \textit{left}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad (\wedge\text{: } \textit{right})$$

$$\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C} \quad (\vee\text{: } \textit{left})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad (\vee\text{: } \textit{right}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \quad (\vee\text{: } \textit{right})$$

$$\frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} \quad (\supset\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\text{: } \textit{right})$$

There is a close relationship between the natural deduction system $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ and the Gentzen system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$. In fact, there is a procedure $\mathcal{N}$ for translating every proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ into a deduction in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$. The procedure $\mathcal{N}$ has the remarkable property that $\mathcal{N}(\Pi)$ is a deduction in normal form for every proof Π. Since there are deductions in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ that are not in normal form, the function $\mathcal{N}$ is *not* surjective. The situation can be repaired by adding a new rule to $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$, the *cut rule*. Then, there is a procedure $\mathcal{N}$ mapping every proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ to a deduction in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$, and a procedure $\mathcal{G}$ mapping every deduction in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ to a proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot,\boldsymbol{cut}}$.

In order to close the loop, we would need to show that every proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot,\boldsymbol{cut}}$ can be transformed into a proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$, that is, a cut-free proof. It is an extremely interesting and deep fact that the system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot,\boldsymbol{cut}}$ and the system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ are indeed equivalent. This fundamental result known as the *cut elimination theorem* was first proved by Gentzen in 1935 [3]. The proof actually gives an algorithm for converting a proof with cuts into a cut-free proof. The main difficulty is to prove that this algorithm terminates. Gentzen used a fairly complex induction measure which was later simplified by Tait [18].

The contraction rule plays a crucial role in the proof of this theorem, and it is therefore natural to believe that this rule cannot be dispensed with. This is indeed true for the intuitionistic system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ (but it can be dispensed with in the classical system $\mathcal{G}_c^{\supset,\wedge,\vee,\bot}$). If we delete the contraction rule from the system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ (or $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$), certain formulae are no longer provable. For example, $\vdash \neg\neg(P \vee \neg P)$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$, but it is impossible to build a cut-free proof for it without using (*contrac*: *left*). Indeed, the only way to build a cut-free proof for $\vdash \neg\neg(P \vee \neg P)$ without using (*contrac*: *left*) is to proceed as follows:

$$\dfrac{\dfrac{\vdash P \vee \neg P}{\neg(P \vee \neg P) \vdash}}{\vdash \neg\neg(P \vee \neg P)}$$

Since the only rules that could yield a cut-free proof of $\vdash P \vee \neg P$ are the ($\vee$: *right*) rules and neither $\vdash P$ nor $\vdash \neg P$ is provable, it is clear that there is no cut-free proof of $\vdash P \vee \neg P$.

However, $\vdash \neg\neg(P \vee \neg P)$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$, as shown by the following proof (the same example can be worked out in $\mathcal{G}_i^{\supset,\wedge,\vee,\perp}$):

Example 4.1

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \vdash P}{P \vdash P \vee \neg P}}{P, \neg(P \vee \neg P) \vdash}}{\neg(P \vee \neg P) \vdash \neg P}}{\neg(P \vee \neg P) \vdash P \vee \neg P}}{\neg(P \vee \neg P), \neg(P \vee \neg P) \vdash}}{\neg(P \vee \neg P) \vdash}}{\vdash \neg\neg(P \vee \neg P)} \quad (\textit{contrac}\text{: } \textit{left})$$

Nevertheless, it is possible to formulate a cut-free system $\mathcal{GK}_i^{\supset,\wedge,\vee,\perp}$ which is equivalent to $\mathcal{G}_i^{\supset,\wedge,\vee,\perp}$. Such a system due to Kleene [11] has no contraction rule, and the premise of every sequent can be interpreted as a set as opposed to a multiset (Recent results of Lincoln, Scedrov, and Shankar [12], show that in the case of propositional intuitionistic logic restricted to implications, it is possible to formulate a contraction-free system which easily yields the decidability of provability).

5 Definition of the Transformation $\mathcal{N}$ from $\mathcal{G}_i$ to $\mathcal{N}_i$

The purpose of this section is to give a procedure $\mathcal{N}$ mapping every proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\perp}$ to a deduction in $\mathcal{N}_i^{\supset,\wedge,\vee,\perp}$. The procedure $\mathcal{N}$ is defined by induction on the structure of proof trees and requires some preliminary definitions.

Definition 9 *A proof tree* Π *with root node* $\Gamma \vdash C$ *is denoted as*

$$\begin{array}{c}\Pi\\ \Gamma \vdash C\end{array}$$

and similarly a deduction $\mathcal{D}$ with root node $\Gamma \vdash C$ is denoted as

$$\begin{array}{c} \mathcal{D} \\ \Gamma \vdash C \end{array}$$

A proof tree Π whose last inference is

$$\frac{\Gamma \vdash B}{\Delta \vdash D}$$

is denoted as

$$\begin{array}{c} \Pi_1 \\ \dfrac{\Gamma \vdash B}{\Delta \vdash D} \end{array}$$

where Π_1 is the immediate subproof of Π whose root is $\Gamma \vdash B$, and a proof tree Π whose last inference is

$$\frac{\Gamma \vdash B \quad \Gamma \vdash C}{\Delta \vdash D}$$

is denoted as

$$\frac{\begin{array}{cc} \Pi_1 & \Pi_2 \\ \Gamma \vdash B & \Gamma \vdash C \end{array}}{\Delta \vdash D}$$

where Π_1 and Π_2 are the immediate subproofs of Π whose roots are $\Gamma \vdash B$ and $\Gamma \vdash C$ respectively. The same notation applies to deductions.

Given a proof tree Π with root node $\Gamma \vdash C$,

$$\begin{array}{c} \Pi \\ \Gamma \vdash C \end{array}$$

$\mathcal{N}$ yields a deduction $\mathcal{N}(\Pi)$ of C from the set of assumptions Γ^+,

$$\begin{array}{c} \mathcal{N}(\Pi) \\ \Gamma^+ \vdash C \end{array}$$

where Γ^+ is obtained from the multiset Γ. However, one has to exercise some care in defining Γ^+ so that $\mathcal{N}$ is indeed a function. This can be achieved as follows. We can assume that we have a fixed total order $\leq_p$ on the set of all propositions so that they can be enumerated as $P_1, P_2, \ldots$, and a fixed total order $\leq_v$ on the set of all variables so that they can be enumerated as $x_1, x_2, \ldots$

Definition 10 *Given a multiset* $\Gamma = A_1, \ldots, A_n$, *since* $\{A_1, \ldots, A_n\} = \{P_{i_1}, \ldots, P_{i_n}\}$ *where* $P_{i_1} \leq_p P_{i_2} \leq_p \ldots \leq_p P_{i_n}$ *(where* $P_1, P_2, \ldots$, *is the enumeration of all propositions and where* $i_j = i_{j+1}$ *is possible since* Γ *is a multiset), we define* Γ^+ *as* $\Gamma^+ = x_1 : P_{i_1}, \ldots, x_n : P_{i_n}$.

We will also need the following concepts and notation.

Definition 11 *Given a deduction*

$$\begin{array}{c} \mathcal{D} \\ \Gamma \vdash C \end{array}$$

the deduction obtained by adding the additional assumptions Δ *to the lefthand side of every sequent of* $\mathcal{D}$ *is denoted as* $\Delta + \mathcal{D}$, *and it is only well defined provided that* $\mathrm{dom}(\Gamma') \cap \mathrm{dom}(\Delta) = \emptyset$ *for every sequent* $\Gamma' \vdash A$ *occurring in* $\mathcal{D}$. *Similarly, given a sequential proof*

$$\begin{array}{c} \Pi \\ \Gamma \vdash \Delta \end{array}$$

we define the proof $\Lambda + \Pi$ *by adding* Λ *to the lefthand side of every sequent of* Π, *and we define the proof* $\Pi + \Theta$ *by adding* Θ *to the righthand side of every sequent of* Π.

We also need a systematic way of renaming the variables in a deduction.

Definition 12 *Given a deduction* $\mathcal{D}$ *with root node* $\Delta \vdash C$ *the deduction* $\mathcal{D}'$ *obtained from* $\mathcal{D}$ *by rectification is defined inductively as follows:*

If $\mathcal{D}$ *consists of the single node* $y_1 : A_1, \ldots, y_m : A_m \vdash C$, *define the total order* $<$ *on the context* $\Delta = y_1 : A_1, \ldots, y_m : A_m$ *as follows:*

$$y_i : A_i < y_j : A_j \quad \textit{iff} \quad \begin{cases} A_i <_p A_j, & \textit{or} \\ A_i = A_j \quad \textit{and} \quad y_i <_v y_j. \end{cases}$$

The order $<$ *on* $y_1 : A_1, \ldots, y_m : A_m$ *defines the permutation* σ *such that*

$$y_{\sigma(1)} : A_{\sigma(1)} < y_{\sigma(2)} : A_{\sigma(2)} < \ldots < y_{\sigma(m-1)} : A_{\sigma(m-1)} < y_{\sigma(m)} : A_{\sigma(m)}.$$

Let $\Delta' = x_1 : A_{\sigma(1)}, \ldots, x_m : A_{\sigma(m)}$, *and define* $\mathcal{D}'$ *as* $\Delta' \vdash C$. *The permutation* σ *induces a bijection between* $\{x_1, \ldots, x_n\}$ *and* $\{y_1, \ldots, y_n\}$, *namely* $x_i \mapsto y_{\sigma(i)}$.

If $\mathcal{D}$ *is of the form*

$$\dfrac{\begin{array}{c} \mathcal{D}_1 \\ y_1 : A_1, y_2 : A_2, \ldots, y_m : A_m \vdash B \end{array}}{y_2 : A_2, \ldots, y_m : A_m \vdash A_1 \supset B}$$

by induction, we have the rectified deduction

$$\begin{array}{c}\mathcal{D}_1'\\ x_1\colon A_{\sigma(1)}, \ldots, x_{j-1}\colon A_{\sigma(j-1)}, x_j\colon A_1, x_{j+1}\colon A_{\sigma(j+1)}, \ldots, x_m\colon A_{\sigma(m)} \vdash B\end{array}$$

where x_j corresponds to y_1 in the bijection between $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ (in fact, $j = \sigma^{-1}(1)$ since $A_1 = A_{\sigma(j)}$). Then, apply the substitution $[x_m/x_j, x_j/x_{j+1}, \ldots, x_{m-1}/x_m]$ to the deduction $\mathcal{D}_1'$, and form the deduction

$$\frac{\begin{array}{c}\mathcal{D}_1'[x_m/x_j, x_j/x_{j+1}, \ldots, x_{m-1}/x_m]\\ x_1\colon A_{\sigma(1)}, \ldots, x_{j-1}\colon A_{\sigma(j-1)}, x_m\colon A_1, x_j\colon A_{\sigma(j+1)}, \ldots, x_{m-1}\colon A_{\sigma(m)} \vdash B\end{array}}{x_1\colon A_{\sigma(1)}, \ldots, x_{j-1}\colon A_{\sigma(j-1)}, x_j\colon A_{\sigma(j+1)}, \ldots, x_{m-1}\colon A_{\sigma(m)} \vdash A_1 \supset B}$$

The other inference rules do not modify the lefthand side of sequents, and $\mathcal{D}'$ is obtained by rectifying the immediate subtree(s) of $\mathcal{D}$.

Note that for any deduction $\mathcal{D}$ with root node $y_1\colon A_1, \ldots, y_m\colon A_m \vdash C$, the rectified deduction $\mathcal{D}'$ has for its root node the sequent $\Gamma^+ \vdash C$, where Γ^+ is obtained from the multiset $\Gamma = A_1, \ldots, A_m$ as in Definition 10.

The procedure $\mathcal{N}$ is defined by induction on the structure of the proof tree Π.

- An axiom $\Gamma, A \vdash A$ is mapped to the deduction $(\Gamma, A)^+ \vdash A$.

- A proof Π of the form

$$\frac{\begin{array}{c}\Pi_1\\ \Gamma \vdash \bot\end{array}}{\Gamma \vdash A}$$

is mapped to the deduction

$$\frac{\begin{array}{c}\mathcal{N}(\Pi_1)\\ \Gamma^+ \vdash \bot\end{array}}{\Gamma^+ \vdash A}$$

- A proof Π of the form

$$\frac{\begin{array}{c}\Pi_1\\ A, A, \Gamma \vdash B\end{array}}{A, \Gamma \vdash B}$$

is mapped to a deduction as follows. First map Π_1 to the deduction $\mathcal{N}(\Pi_1)$

$$\begin{array}{c}\mathcal{N}(\Pi_1)\\ x\colon A, y\colon A, \Gamma^+ \vdash B\end{array}$$

Next, replace every occurrence of "$x\colon A, y\colon A$" in $\mathcal{N}(\Pi_1)$ by "$z\colon A$" where z is a new variable not occurring in $\mathcal{N}(\Pi_1)$, and finally rectify the resulting tree.

• A proof Π of the form

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma \vdash A & \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B}$$

is mapped to the deduction

$$\dfrac{\begin{array}{cc}\mathcal{N}(\Pi_1) & \mathcal{N}(\Pi_2)\\ \Gamma^+ \vdash A & \Gamma^+ \vdash B\end{array}}{\Gamma^+ \vdash A \wedge B}$$

• A proof Π of the form

$$\dfrac{\begin{array}{c}\Pi_1\\ A, B, \Gamma \vdash C\end{array}}{A \wedge B, \Gamma \vdash C}$$

is mapped to a deduction obtained as follows. First, map Π_1 to $\mathcal{N}(\Pi_1)$

$$\begin{array}{c}\mathcal{N}(\Pi_1)\\ x\colon A, y\colon B, \Gamma^+ \vdash C\end{array}$$

Next, replace every leaf of the form $x\colon A, y\colon B, \Delta, \Gamma^+ \vdash A$ in $\mathcal{N}(\Pi_1)$ by the subtree

$$\dfrac{z\colon A \wedge B, \Delta, \Gamma^+ \vdash A \wedge B}{z\colon A \wedge B, \Delta, \Gamma^+ \vdash A}$$

and every leaf of the form $x\colon A, y\colon B, \Delta, \Gamma^+ \vdash B$ in $\mathcal{N}(\Pi_1)$ by the subtree

$$\dfrac{z\colon A \wedge B, \Delta, \Gamma^+ \vdash A \wedge B}{z\colon A \wedge B, \Delta, \Gamma^+ \vdash B}$$

where z is new, replace "$x\colon A, y\colon B$" by "$z\colon A \wedge B$" in every antecedent of the resulting deduction, and rectify this last tree.

• A proof Π of the form

$$\dfrac{\begin{array}{c}\Pi_1\\ A, \Gamma \vdash B\end{array}}{\Gamma \vdash A \supset B}$$

is mapped to the deduction

$$\dfrac{\begin{array}{c}\mathcal{N}(\Pi_1)\\ x\colon A, \Gamma^+ \vdash B\end{array}}{\Gamma^+ \vdash A \supset B}$$

• A proof Π of the form

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma \vdash A & B, \Gamma \vdash C\end{array}}{A \supset B, \Gamma \vdash C}$$

is mapped to a deduction as follows. First map Π_1 and Π_2 to deductions $\mathcal{N}(\Pi_1)$

$$\begin{array}{c}\mathcal{N}(\Pi_1)\\ \Gamma^+ \vdash A\end{array}$$

and $\mathcal{N}(\Pi_2)$

$$\begin{array}{c}\mathcal{N}(\Pi_2)\\ x\colon B, \Gamma^+ \vdash C\end{array}$$

Next, form the deduction $\mathcal{D}$

$$\dfrac{\begin{array}{cc} & z\colon A \supset B + \mathcal{N}(\Pi_1)\\ z\colon A \supset B, \Gamma^+ \vdash A \supset B & z\colon A \supset B, \Gamma^+ \vdash A\end{array}}{z\colon A \supset B, \Gamma^+ \vdash B}$$

and modify $\mathcal{N}(\Pi_2)$ as follows: replace every leaf of the form $x\colon B, \Delta, \Gamma^+ \vdash B$ by the deduction obtained from $\Delta + \mathcal{D}$ by replacing "$x\colon B$" by "$z\colon A \supset B$" in the lefthand side of every sequent. Finally, rectify this last deduction.

• A proof Π of the form

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \vee B}$$

is mapped to the deduction

$$\dfrac{\begin{array}{c}\mathcal{N}(\Pi_1)\\ \Gamma^+ \vdash A\end{array}}{\Gamma^+ \vdash A \vee B}$$

and similarly for the other case of the ($\vee$: *right*) rule.

• A proof Π of the form

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ A, \Gamma \vdash C & B, \Gamma \vdash C\end{array}}{A \vee B, \Gamma \vdash C}$$

is mapped to a deduction as follows. First map Π_1 and Π_2 to deductions $\mathcal{N}(\Pi_1)$

$$\begin{array}{c}\mathcal{N}(\Pi_1)\\ x\colon A, \Gamma^+ \vdash C\end{array}$$

and $\mathcal{N}(\Pi_2)$

$$\begin{array}{c}\mathcal{N}(\Pi_2)\\ y\colon B, \Gamma^+ \vdash C\end{array}$$

Next, form the deduction

$$\dfrac{\begin{array}{ccc} & z\colon A \vee B + \mathcal{N}(\Pi_1) & z\colon A \vee B + \mathcal{N}(\Pi_2)\\ z\colon A \vee B, \Gamma^+ \vdash A \vee B & z\colon A \vee B, x\colon A, \Gamma^+ \vdash C & z\colon A \vee B, y\colon B, \Gamma^+ \vdash C\end{array}}{z\colon A \vee B, \Gamma^+ \vdash C}$$

and rectify this last tree.

This concludes the definition of the procedure $\mathcal{N}$. Note that the contraction rule can be stated in the system of natural deduction as follows:

$$\dfrac{x\colon A, y\colon A, \Gamma \vdash B}{z\colon A, \Gamma \vdash B[z/x, z/y]}$$

where z is a new variable. The following remarkable property of $\mathcal{N}$ is easily shown.

Lemma 1 (Gentzen (1935), Prawitz (1965)) *For every proof* Π *in* $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$, $\mathcal{N}(\Pi)$ *is a deduction in normal form (in* $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$*).*

Since there are deductions in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ that are not in normal form, the function $\mathcal{N}$ is not surjective. It is interesting to observe that the function $\mathcal{N}$ is not injective either. What happens is that $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ is more sequential than $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$, in the sense that the order of application of inferences is strictly recorded. Hence, two proofs in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ of the same sequent may differ for bureaucratic reasons: independent inferences are applied in different orders. In $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$, these differences disappear. The following example illustrates this point. The sequent $\vdash (P \wedge P') \supset ((Q \wedge Q') \supset (P \wedge Q))$ has the following two sequential proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P, P', Q, Q' \vdash P \qquad P, P', Q, Q' \vdash Q}{P, P', Q, Q' \vdash P \wedge Q}}{P \wedge P', Q, Q' \vdash P \wedge Q}}{P \wedge P', Q \wedge Q' \vdash P \wedge Q}}{P \wedge P' \vdash (Q \wedge Q') \supset (P \wedge Q)}}{\vdash (P \wedge P') \supset ((Q \wedge Q') \supset (P \wedge Q))}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P, P', Q, Q' \vdash P \qquad P, P', Q, Q' \vdash Q}{P, P', Q, Q' \vdash P \wedge Q}}{P, P', Q \wedge Q' \vdash P \wedge Q}}{P \wedge P', Q \wedge Q' \vdash P \wedge Q}}{P \wedge P' \vdash (Q \wedge Q') \supset (P \wedge Q)}}{\vdash (P \wedge P') \supset ((Q \wedge Q') \supset (P \wedge Q))}$$

Both proofs are mapped to the deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{x\colon P \wedge P', y\colon Q \wedge Q' \vdash P \wedge P'}{x\colon P \wedge P', y\colon Q \wedge Q' \vdash P} \qquad \dfrac{x\colon P \wedge P', y\colon Q \wedge Q' \vdash Q \wedge Q'}{x\colon P \wedge P', y\colon Q \wedge Q' \vdash Q}}{x\colon P \wedge P', y\colon Q \wedge Q' \vdash P \wedge Q}}{x\colon P \wedge P' \vdash (Q \wedge Q') \supset (P \wedge Q)}}{\vdash (P \wedge P') \supset ((Q \wedge Q') \supset (P \wedge Q))}$$

6 Definition of the Transformation $\mathcal{G}$ from $\mathcal{N}_i$ to $\mathcal{G}_i$

We now show that if we add a new rule, the cut rule, to the system $\mathcal{G}_i^{\supset,\wedge,\vee,\perp}$, then we can define a procedure $\mathcal{G}$ mapping every deduction in $\mathcal{N}_i^{\supset,\wedge,\vee,\perp}$ to a proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\perp,cut}$.

Definition 13 *The system $\mathcal{G}_i^{\supset,\wedge,\vee,\perp,cut}$ is obtained from the system $\mathcal{G}_i^{\supset,\wedge,\vee,\perp}$ by adding the following rule, known as the cut rule:*

$$\frac{\Gamma \vdash A \quad A, \Gamma \vdash C}{\Gamma \vdash C} \quad \textit{(cut)}$$

The system $\mathcal{G}_c^{\supset,\wedge,\vee,\perp,cut}$ is obtained from $\mathcal{G}_c^{\supset,\wedge,\vee,\perp}$ by adding the following rule, also known as the cut rule:

$$\frac{\Gamma \vdash A, \Delta \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \quad \textit{(cut)}$$

Next, we define the procedure $\mathcal{G}$ mapping every deduction in $\mathcal{N}_i^{\supset,\wedge,\vee,\perp}$ to a proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\perp,cut}$. The procedure $\mathcal{G}$ is defined by induction on the structure of deduction trees. Given a deduction tree $\mathcal{D}$ of C from the assumptions Γ,

$$\begin{array}{c} \mathcal{D} \\ \Gamma \vdash C \end{array}$$

$\mathcal{G}$ yields a proof $\mathcal{G}(\mathcal{D})$ of the sequent $\Gamma^- \vdash C$

$$\begin{array}{c} \mathcal{G}(\mathcal{D}) \\ \Gamma^- \vdash C \end{array}$$

where Γ^- is the multiset $A_1, \ldots, A_n$ obtained from the context $\Gamma = x_1\colon A_1, \ldots, x_n\colon A_n$ by erasing $x_1, \ldots, x_n$, where $x_1, \ldots, x_n$ are pairwise distinct.

- The deduction $\Gamma, x\colon A \vdash A$ is mapped to the axiom $\Gamma^-, A \vdash A$.

- A deduction $\mathcal{D}$ of the form

$$\frac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \vdash \perp\end{array}}{\Gamma \vdash A}$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{G}(\mathcal{D}_1)\\ \Gamma^- \vdash \bot\end{array}}{\Gamma^- \vdash A}$$

• A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B}$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{G}(\mathcal{D}_1)\\ \Gamma^- \vdash A\end{array} \qquad \begin{array}{c}\mathcal{G}(\mathcal{D}_2)\\ \Gamma^- \vdash B\end{array}}{\Gamma^- \vdash A \wedge B}$$

• A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A \wedge B\end{array}}{\Gamma \vdash A}$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{G}(\mathcal{D}_1)\\ \Gamma^- \vdash A \wedge B\end{array} \qquad \dfrac{A, B, \Gamma^- \vdash A}{A \wedge B, \Gamma^- \vdash A}}{\Gamma^- \vdash A}\ (\mathit{cut})$$

and similarly for the symmetric rule.

• A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ x\colon A, \Gamma \vdash B\end{array}}{\Gamma \vdash A \supset B}$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{G}(\mathcal{D}_1)\\ A, \Gamma^- \vdash B\end{array}}{\Gamma^- \vdash A \supset B}$$

• A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \Gamma \vdash A \supset B & \Gamma \vdash A\end{array}}{\Gamma \vdash B}$$

is mapped to the proof

$$\dfrac{\begin{array}{cc}\mathcal{G}(\mathcal{D}_1) & \dfrac{\begin{array}{cc}\mathcal{G}(\mathcal{D}_2) & \\ \Gamma^- \vdash A & B, \Gamma^- \vdash B\end{array}}{A \supset B, \Gamma^- \vdash B}\\ \Gamma^- \vdash A \supset B & \end{array}}{\Gamma^- \vdash B}\ (\mathit{cut})$$

• A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \vee B}$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{G}(\mathcal{D}_1)\\ \Gamma^- \vdash A\end{array}}{\Gamma^- \vdash A \vee B}$$

and similarly for the symmetric rule.

• A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{ccc}\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3\\ \Gamma \vdash A \vee B & x\colon A, \Gamma \vdash C & y\colon B, \Gamma \vdash C\end{array}}{\Gamma \vdash C}$$

is mapped to the proof

$$\dfrac{\begin{array}{cc}\mathcal{G}(\mathcal{D}_1) & \dfrac{\begin{array}{cc}\mathcal{G}(\mathcal{D}_2) & \mathcal{G}(\mathcal{D}_3)\\ A, \Gamma^- \vdash C & B, \Gamma^- \vdash C\end{array}}{A \vee B, \Gamma^- \vdash C}\\ \Gamma^- \vdash A \vee B & \end{array}}{\Gamma^- \vdash C}\ (\mathit{cut})$$

This concludes the definition of the procedure $\mathcal{G}$.

For the sake of completeness, we also extend the definition of the function $\mathcal{N}$ which is presently defined on the set of sequential proofs of the system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ to proofs with cuts, that is, to proofs in the system $\mathcal{N}_i^{\supset,\wedge,\vee,\bot,cut}$. A proof Π of the form

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma \vdash A & A, \Gamma \vdash C\end{array}}{\Gamma \vdash C}$$

is mapped to the deduction obtained as follows: First, construct

$$\begin{array}{c}\mathcal{N}(\Pi_1)\\ \Gamma^+ \vdash A\end{array}$$

and $\mathcal{N}(\Pi_2)$

$$\begin{array}{c}\mathcal{N}(\Pi_2)\\ x\colon A, \Gamma^+ \vdash C\end{array}$$

Then, replace every leaf $x\colon A, \Delta, \Gamma^+ \vdash A$ in $\mathcal{N}(\Pi_2)$ by $\Delta + \mathcal{N}(\Pi_1)$, delete "$x\colon A$" from the antecedent in every sequent, and rectify this last tree.

7 First-Order Quantifiers

We extend the systems $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ and $\mathcal{G}_i^{\supset,\wedge,\vee,\bot,cut}$ to deal with the quantifiers.

Definition 14 *The axioms and inference rules of the system $\mathcal{N}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ for intuitionistic first-order logic are listed below:*

$$\Gamma, x\colon A \vdash A$$

$$\dfrac{\Gamma, x\colon A \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\textit{-intro})$$

$$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \quad (\supset\textit{-elim})$$

$$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad (\wedge\textit{-intro})$$

$$\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \quad (\wedge\textit{-elim}) \qquad \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \quad (\wedge\textit{-elim})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad (\vee\text{-intro}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \quad (\vee\text{-intro})$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, x\colon A \vdash C \quad \Gamma, y\colon B \vdash C}{\Gamma \vdash C} \quad (\vee\text{-elim})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \quad (\bot\text{-elim})$$

$$\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A} \quad (\forall\text{-intro}) \qquad \frac{\Gamma \vdash \forall x A}{\Gamma \vdash A[\tau/x]} \quad (\forall\text{-elim})$$

where in ($\forall$-intro), y does not occur free in Γ or $\forall x A$;

$$\frac{\Gamma \vdash A[\tau/x]}{\Gamma \vdash \exists x A} \quad (\exists\text{-intro}) \qquad \frac{\Gamma \vdash \exists x A \quad z\colon A[y/x], \Gamma \vdash C}{\Gamma \vdash C} \quad (\exists\text{-elim})$$

where in ($\exists$-elim), y does not occur free in Γ, $\exists x A$, or C.

The variable y is called the eigenvariable of the inference.

Definition 15 *The axioms and inference rules of the system $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot,cut}$ for intuitionistic first-order logic are given below.*

$$A, \Gamma \vdash A$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \quad (\bot\text{: right})$$

$$\frac{A, A, \Gamma \vdash C}{A, \Gamma \vdash C} \quad (\textit{contrac}\text{: left})$$

$$\frac{\Gamma \vdash A \quad A, \Gamma \vdash C}{\Gamma \vdash C} \quad (cut)$$

$$\frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C} \quad (\wedge\text{: left}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad (\wedge\text{: right})$$

$$\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C} \quad (\vee\text{: left})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad (\vee\text{: right}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \quad (\vee\text{: right})$$

$$\frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} \quad (\supset\text{: left}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\text{: right})$$

$$\frac{A[\tau/x], \Gamma \vdash C}{\forall x A, \Gamma \vdash C} \quad (\forall\text{: left)} \qquad \frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A} \quad (\forall\text{: right)}$$

where in ($\forall$*: right),* y *does not occur free in the conclusion;*

$$\frac{A[y/x], \Gamma \vdash C}{\exists x A, \Gamma \vdash C} \quad (\exists\text{: left)} \qquad \frac{\Gamma \vdash A[\tau/x]}{\Gamma \vdash \exists x A} \quad (\exists\text{: right)}$$

where in ($\exists$*: left),* y *does not occur free in the conclusion.*

The variable y *is called the eigenvariable of the inference.*

The typed λ-calculus $\lambda^{\rightarrow,\times,+,\forall,\exists,\perp}$ corresponding to $\mathcal{N}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$ is given in the following definition.

Definition 16 *The typed* λ*-calculus* $\lambda^{\rightarrow,\times,+,\forall,\exists,\perp}$ *is defined by the following rules.*

$$\Gamma, x\colon A \triangleright x\colon A$$

$$\frac{\Gamma, x\colon A \triangleright M\colon B}{\Gamma \triangleright (\lambda x\colon A.\ M)\colon A \to B} \quad \textit{(abstraction)}$$

$$\frac{\Gamma \triangleright M\colon A \to B \quad \Gamma \triangleright N\colon A}{\Gamma \triangleright (M N)\colon B} \quad \textit{(application)}$$

$$\frac{\Gamma \triangleright M\colon A \quad \Gamma \triangleright N\colon B}{\Gamma \triangleright \langle M, N\rangle\colon A \times B} \quad \textit{(pairing)}$$

$$\frac{\Gamma \triangleright M\colon A \times B}{\Gamma \triangleright \pi_1(M)\colon A} \quad \textit{(projection)} \qquad \frac{\Gamma \triangleright M\colon A \times B}{\Gamma \triangleright \pi_2(M)\colon B} \quad \textit{(projection)}$$

$$\frac{\Gamma \triangleright M\colon A}{\Gamma \triangleright \mathtt{inl}(M)\colon A + B} \quad \textit{(injection)} \qquad \frac{\Gamma \triangleright M\colon B}{\Gamma \triangleright \mathtt{inr}(M)\colon A + B} \quad \textit{(injection)}$$

$$\frac{\Gamma \triangleright P\colon A + B \quad \Gamma, x\colon A \triangleright M\colon C \quad \Gamma, y\colon B \triangleright N\colon C}{\Gamma \triangleright \mathtt{case}(P, \lambda x\colon A.\ M, \lambda y\colon B.\ N)\colon C} \quad \textit{(by-cases)}$$

or

$$\frac{\Gamma \triangleright P\colon A + B \quad \Gamma, x\colon A \triangleright M\colon C \quad \Gamma, y\colon B \triangleright N\colon C}{\Gamma \triangleright (\mathtt{case}\ P\ \mathtt{of}\ \mathtt{inl}(x\colon A) \Rightarrow M \mid \mathtt{inr}(y\colon B) \Rightarrow N)\colon C} \quad \textit{(by-cases)}$$

$$\frac{\Gamma \triangleright M\colon \perp}{\Gamma \triangleright \triangle_A(M)\colon A} \quad (\perp\textit{-elim)}$$

$$\frac{\Gamma \triangleright M\colon A[u/t]}{\Gamma \triangleright (\lambda t\colon \iota.\ M)\colon \forall t A} \quad (\forall\textit{-intro)}$$

where u does not occur free in Γ or $\forall t A$;

$$\frac{\Gamma \triangleright M : \forall t A}{\Gamma \triangleright M\tau : A[\tau/t]} \quad (\forall\text{-elim})$$

$$\frac{\Gamma \triangleright M : A[\tau/t]}{\Gamma \triangleright \mathtt{pair}(\tau, M) : \exists t A} \quad (\exists\text{-intro})$$

$$\frac{\Gamma \triangleright M : \exists t A \quad \Gamma, x : A[u/t] \triangleright N : C}{\Gamma \triangleright \mathtt{select}(M, \lambda t{:}\iota.\ \lambda x{:}A.\ N) : C} \quad (\exists\text{-elim})$$

where u does not occur free in Γ, $\exists t A$, or C.

In the term $(\lambda t{:}\iota.\ M)$, the type ι stands for the type of individuals. Note that $\Gamma \triangleright \lambda t{:}\iota.\ \lambda x{:}A.\ N : \forall t(A \to C)$. The term $\lambda t{:}\iota.\ \lambda x{:}A.\ N$ contains the type A which is a *dependent* type, since it usually contains occurrences of t. Observe that $(\lambda t{:}\iota.\ \lambda x{:}A.\ N)\tau$ reduces to $\lambda x{:}A[\tau/t].\ N[\tau/t]$, in which the type of x is now $A[\tau/t]$. The term $\mathtt{select}(M, \lambda t{:}\iota.\ \lambda x{:}A.\ N)$ is also denoted as $\mathtt{select}\ M\ \mathtt{of}\ \mathtt{pair}(t{:}\iota, x{:}A) \Rightarrow N$, or even $\mathtt{select}\ M\ \mathtt{of}\ \mathtt{pair}(t, x) \Rightarrow N$, and the ($\exists$-*elim*) rule as

$$\frac{\Gamma \triangleright M : \exists t A \quad \Gamma, x : A[u/t] \triangleright N : C}{\Gamma \triangleright (\mathtt{select}\ M\ \mathtt{of}\ \mathtt{pair}(t{:}\iota, x{:}A) \Rightarrow N) : C} \quad (\exists\text{-elim})$$

where u does not occur free in Γ, $\exists t A$, or C.

Such a formalism can be easily generalized to many sorts (base types), if quantified formulae are written as $\forall t{:}\sigma.\ A$ and $\exists t{:}\sigma.\ A$, where σ is a sort (base type). We also have the following reduction rules.

Definition 17 *The reduction rules of the system $\lambda^{\to,\times,+,\forall,\exists,\perp}$ are listed below:*

$$\begin{aligned}
(\lambda x{:}A.\ M)N &\longrightarrow M[N/x],\\
\pi_1(\langle M, N\rangle) &\longrightarrow M,\\
\pi_2(\langle M, N\rangle) &\longrightarrow N,\\
\mathtt{case}(\mathtt{inl}(P), M, N) &\longrightarrow MP, \quad \textit{or}\\
\mathtt{case}\ \mathtt{inl}(P)\ \mathtt{of}\ \mathtt{inl}(x{:}A) \Rightarrow M \mid \mathtt{inr}(y{:}B) \Rightarrow N &\longrightarrow M[P/x],\\
\mathtt{case}(\mathtt{inr}(P), M, N) &\longrightarrow NP, \quad \textit{or}\\
\mathtt{case}\ \mathtt{inr}(P)\ \mathtt{of}\ \mathtt{inl}(x{:}A) \Rightarrow M \mid \mathtt{inr}(y{:}B) \Rightarrow N &\longrightarrow N[P/y],\\
\triangle_{A\to B}(M)N &\longrightarrow \triangle_B(M),\\
\pi_1(\triangle_{A\times B}(M)) &\longrightarrow \triangle_A(M),\\
\pi_2(\triangle_{A\times B}(M)) &\longrightarrow \triangle_B(M),\\
(\lambda t{:}\iota.\ M)\tau &\longrightarrow M[\tau/t],
\end{aligned}$$

$$\begin{aligned}
\triangle_{\forall tA}(M)\tau &\longrightarrow \triangle_{A[\tau/t]}(M),\\
\texttt{case}(\triangle_{A+B}(P), M, N) &\longrightarrow \triangle_C(P),\\
\texttt{select}(\texttt{pair}(\tau, P), M) &\longrightarrow (M\tau)P, \quad \textit{or}\\
\texttt{select pair}(\tau, P)\ \texttt{of pair}(t{:}\iota, x{:}A) \Rightarrow N &\longrightarrow N[\tau/t, P/x],\\
\texttt{select}(\triangle_{\exists tA}(P), M) &\longrightarrow \triangle_C(P),\\
\texttt{select}(\texttt{pair}(\tau, P), \triangle_{\forall t(A\to C)}(M)) &\longrightarrow \triangle_{C[\tau/t]}(M),\\
\triangle_A(\triangle_\bot(M)) &\longrightarrow \triangle_A(M).
\end{aligned}$$

A fundamental result about natural deduction is the fact that every proof (term) reduces to a normal form, which is unique up to α-renaming. This result was first proved by Prawitz [15] for the system $\mathcal{N}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$.

Theorem 3 (Church-Rosser property, Prawitz (1971)) *Reduction in $\lambda^{\to,\times,+,\forall,\exists,\bot}$ (specified in Definition 17) is confluent. Equivalently, conversion in $\lambda^{\to,\times,+,\forall,\exists,\bot}$ is Church-Rosser.*

A proof can be given by adapting the method of Tait and Martin-Löf [13] using a form of parallel reduction (see also Stenlund [16]).

Theorem 4 (Strong normalization property, Prawitz (1971)) *Reduction in $\lambda^{\to,\times,+,\forall,\exists,\bot}$ is strongly normalizing.*

A proof can be given by adapting Tait's reducibility method [17], [19], as done in Girard [5] (1971), [6] (1972) (see also Gallier [2]).

To obtain the system $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\bot,cut}$ of classical logic, we add to $\mathcal{G}_c^{\supset,\wedge,\vee,\bot}$ the cut rule and the quantifier rules shown in the next definition.

Definition 18 *The axioms and inference rules of the system $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\bot,cut}$ for classical first-order logic are given below.*

$$A, \Gamma \vdash \Delta, A$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \textit{(contrac: left)} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \textit{(contrac: right)}$$

$$\frac{\Gamma \vdash \Delta, \bot}{\Gamma \vdash \Delta, A}\ (\bot\textit{: right})$$

$$\frac{\Gamma \vdash A, \Delta \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \textit{(cut)}$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ (\wedge\textit{: left}) \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ (\wedge\textit{: right})$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \ (\vee\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \ (\vee\text{: } \textit{right})$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \ (\supset\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B} \ (\supset\text{: } \textit{right})$$

$$\frac{A[\tau/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \ (\forall\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x A} \ (\forall\text{: } \textit{right})$$

where in ($\forall$: *right*), y *does not occur free in the conclusion;*

$$\frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \ (\exists\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A[\tau/x]}{\Gamma \vdash \Delta, \exists x A} \ (\exists\text{: } \textit{right})$$

where in ($\exists$: *left*), y *does not occur free in the conclusion.*

We now extend the functions $\mathcal{N}$ and $\mathcal{G}$ to deal with the quantifier rules. The procedure $\mathcal{N}$ is extended to the quantifier rules as follows.

• A proof Π of the form

$$\frac{\begin{array}{c}\Pi_1 \\ A[\tau/x], \Gamma \vdash C\end{array}}{\forall x A, \Gamma \vdash C}$$

is mapped to a deduction obtained as follows. First, map Π_1 to $\mathcal{N}(\Pi_1)$

$$\begin{array}{c}\mathcal{N}(\Pi_1) \\ y\text{: } A[\tau/x], \Gamma^+ \vdash C\end{array}$$

Next, replace every leaf of the form y: $A[\tau/x], \Delta, \Gamma^+ \vdash A[\tau/x]$ in $\mathcal{N}(\Pi_1)$ by the subtree

$$\frac{y\text{: } \forall x A, \Delta, \Gamma^+ \vdash \forall x A}{y\text{: } \forall x A, \Delta, \Gamma^+ \vdash A[\tau/x]}$$

and rectify this last tree.

• A proof Π of the form

$$\frac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash A[y/x]\end{array}}{\Gamma \vdash \forall x A}$$

is mapped to the deduction

$$\dfrac{\begin{array}{c}\mathcal{N}(\Pi_1)\\ \Gamma^+ \vdash A[y/x]\end{array}}{\Gamma^+ \vdash \forall x\, A}$$

• A proof Π of the form

$$\dfrac{\begin{array}{c}\Pi_1\\ A[y/x], \Gamma \vdash C\end{array}}{\exists x\, A, \Gamma \vdash C}$$

is mapped to the deduction

$$\dfrac{u\colon \exists x\, A, \Gamma^+ \vdash \exists x\, A \qquad \begin{array}{c} \\ u\colon \exists x\, A + \mathcal{N}(\Pi_1) \\ u\colon \exists x\, A, v\colon A[y/x], \Gamma^+ \vdash C\end{array}}{u\colon \exists x\, A, \Gamma^+ \vdash C}$$

and rectify this last tree.

• A proof Π of the form

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A[\tau/x]\end{array}}{\Gamma \vdash \exists x\, A}$$

is mapped to the deduction

$$\dfrac{\begin{array}{c}\mathcal{N}(\Pi_1)\\ \Gamma^+ \vdash A[\tau/x]\end{array}}{\Gamma^+ \vdash \exists x\, A}$$

It is easily seen that Lemma 1 generalizes to quantifiers.

Lemma 2 (Gentzen (1935), Prawitz (1965)) *For every proof* Π *in* $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$, $\mathcal{N}(\Pi)$ *is a deduction in normal form (in* $\mathcal{N}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$*).*

Next, we extend the procedure $\mathcal{G}$ to deal with the quantifier rules.

• A deduction $\mathcal{D}$ of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A[y/x]\end{array}}{\Gamma \vdash \forall x A}$$

is mapped to the proof

$$\frac{\begin{array}{c}\mathcal{G}(\mathcal{D}_1)\\ \Gamma^- \vdash A[y/x]\end{array}}{\Gamma^- \vdash \forall x A}$$

• A deduction $\mathcal{D}$ of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash \forall x A\end{array}}{\Gamma \vdash A[\tau/x]}$$

is mapped to the proof

$$\frac{\begin{array}{c}\mathcal{G}(\mathcal{D}_1)\\ \Gamma^- \vdash \forall x A\end{array} \qquad \frac{A[\tau/x], \Gamma^- \vdash A[\tau/x]}{\forall x A, \Gamma^- \vdash A[\tau/x]}}{\Gamma^- \vdash A[\tau/x]}\ (\textit{cut})$$

• A deduction $\mathcal{D}$ of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A[\tau/x]\end{array}}{\Gamma \vdash \exists x A}$$

is mapped to the proof

$$\frac{\begin{array}{c}\mathcal{G}(\mathcal{D}_1)\\ \Gamma^- \vdash A[\tau/x]\end{array}}{\Gamma^- \vdash \exists x A}$$

• A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma \vdash \exists x\, A & z\colon A[y/x], \Gamma \vdash C\end{array}}{\Gamma \vdash C}$$

is mapped to the proof

$$\dfrac{\begin{array}{cc}\mathcal{G}(\mathcal{D}_1) & \dfrac{\mathcal{G}(\mathcal{D}_2)}{\dfrac{A[y/x], \Gamma^- \vdash C}{\exists x\, A, \Gamma^- \vdash C}} \\ \Gamma^- \vdash \exists x\, A & \end{array}}{\Gamma^- \vdash C}\ (\mathit{cut})$$

We now turn to cut elimination.

8 Gentzen's Cut Elimination Theorem

As we said earlier before presenting the function $\mathcal{G}$ from $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ to $\mathcal{G}_i^{\supset,\wedge,\vee,\bot,\mathit{cut}}$, it is possible to show that the system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot,\mathit{cut}}$ is equivalent to the seemingly weaker system $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$.

We have the following fundamental result.

Theorem 5 (Cut Elimination Theorem, Gentzen (1935)) *There is an algorithm which, given any proof* Π *in* $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot,\mathit{cut}}$ *produces a cut-free proof* Π' *in* $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$. *There is an algorithm which, given any proof* Π *in* $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\bot,\mathit{cut}}$ *produces a cut-free proof* Π' *in* $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\bot}$.

Proof. The proof is quite involved. It consists in pushing up cuts towards the leaves, and in breaking cuts involving compound formulae into cuts on smaller subformulae. Full details are given for the system $\mathcal{LK}$ in Section 11. Interestingly, the need for the contraction rule arises when a cut involves an axiom. The typical example is as follows. The proof

$$\dfrac{\begin{array}{cc} & \Pi_1 \\ A, \Gamma \vdash A & A, A, \Gamma \vdash C\end{array}}{A, \Gamma \vdash C}$$

is equivalent to a (*contrac*: *left*), and it is eliminated by forming the proof

$$\frac{\begin{array}{c}\Pi_1\\ A, A, \Gamma \vdash C\end{array}}{A, \Gamma \vdash C} \quad (\textit{contrac}\text{: }\textit{left})$$

□

If we are interested in cut-free proofs, except in *classical propositional logic*, the contraction rules *cannot* be dispensed with. We already saw in Example 4.1 that $\neg\neg(P \vee \neg P)$ is a proposition which is not provable without contractions in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$. Another example involving quantifiers is the sequent $\forall x \exists y(Py \wedge \neg Px) \vdash$ which is not provable without contractions in $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\neg}$ or even in $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\neg}$. This sequent has the following proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\neg}$:

Example 8.1

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pu, \neg Px, Pv \vdash Pu}{Pu, \neg Px, Pv, \neg Pu \vdash}}{Pu, \neg Px, (Pv \wedge \neg Pu) \vdash}}{(Pu \wedge \neg Px), (Pv \wedge \neg Pu) \vdash}}{(Pu \wedge \neg Px), \exists y(Py \wedge \neg Pu) \vdash}}{(Pu \wedge \neg Px), \forall x \exists y(Py \wedge \neg Px) \vdash}}{\exists y(Py \wedge \neg Px), \forall x \exists y(Py \wedge \neg Px) \vdash}}{\forall x \exists y(Py \wedge \neg Px), \forall x \exists y(Py \wedge \neg Px) \vdash}}{\forall x \exists y(Py \wedge \neg Px) \vdash} \quad (\textit{contrac}\text{: }\textit{left})$$

It is an interesting exercise to find a deduction of $\forall x \exists y(Py \wedge \neg Px) \supset \bot$ in $\mathcal{N}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$.

For classical logic, it is possible to show that the contraction rules are only needed to permit an unbounded number of applications of the ($\forall$: *left*)-rule and the ($\exists$: *right*)-rule. For example, the formula $\exists x \forall y(Py \supset Px)$ is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\bot}$, but not without the rule (*contrac*: *right*). The system $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\bot}$ can be modified to obtain another system $\mathcal{GK}_c^{\supset,\wedge,\vee,\forall,\exists,\bot}$ in which the contraction rules are deleted and the quantifier rules are as follows:

$$\frac{\forall x A, A[\tau/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \quad (\forall\text{: }\textit{left}) \qquad \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x A} \quad (\forall\text{: }\textit{right})$$

where in ($\forall$: *right*), y does not occur free in the conclusion;

$$\frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \quad (\exists\text{: }\textit{left}) \qquad \frac{\Gamma \vdash \Delta, \exists x A, A[\tau/x]}{\Gamma \vdash \Delta, \exists x A} \quad (\exists\text{: }\textit{right})$$

where in ($\exists$: *left*), y does not occur free in the conclusion.

The above system is inspired from Kleene [11] (see system $G3$, page 481). Note that contraction steps have been incorporated in the ($\forall$: *left*)-rule and the ($\exists$: *right*)-rule. The equivalence of the systems $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\bot}$ and $\mathcal{GK}_c^{\supset,\wedge,\vee,\forall,\exists,\bot}$ is shown using two lemmas inspired from Kleene [11] (1952). Given an inference other than ($\bot$: *right*), note that the inference creates a new occurrence of a formula called the *principal formula*.

Lemma 3 *Given a proof Π in $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\neg}$ of a sequent $\Gamma \vdash \Delta$, for every selected occurrence of a formula of the form $A \wedge B$, $A \vee B$, $A \supset B$, $\neg A$, in Γ or Δ, or $\exists x\, A$, in Γ, or $\forall x\, A$ in Δ, there is another proof Π' whose last inference has the specified occurrence of the formula as its principal formula, and uses no more contractions than Π does.*

Proof. The proof is by induction on the structure of the proof tree. There are a number of cases depending on what the last inference is. □

Lemma 3 does not hold for an occurrence of a formula $\forall x A$ in Γ or for a formula $\exists x A$ in Δ, because the inference that creates it involves a term τ, and moving this inference down in the proof may cause a conflict with the side condition on the eigenvariable y involved in the rules ($\forall$: *right*) or ($\exists$: *left*). As shown by the following example, Lemma 3 also fails for intuitionistic logic. The sequent $P, (P \supset Q), (R \supset S) \vdash Q$ has the following proof:

$$\frac{P, (R \supset S) \vdash P \qquad P, (R \supset S), Q \vdash Q}{P, (P \supset Q), (R \supset S) \vdash Q}$$

On the other hand, the following tree is *not* a proof:

$$\frac{\dfrac{P \vdash P \qquad P, Q \vdash R}{P, (P \supset Q) \vdash R} \qquad \dfrac{P, S \vdash P \qquad P, S, Q \vdash Q}{P, (P \supset Q), S \vdash Q}}{P, (P \supset Q), (R \supset S) \vdash Q}$$

This shows that in searching for a proof, one has to be careful not to stop after the first failure. Since the contraction rule cannot be dispensed with, it is not obvious at all that provability of an intuitionistic propositional sequent is a decidable property. In fact, it is, but proving it requires a fairly subtle argument. We will present an argument due to Kleene. For the time being, we return to classical logic.

Lemma 4 *Given any formula A, any terms τ_1, τ_2, any proof Π in $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\neg}$ of a sequent $A[\tau_1/x], A[\tau_2/x], \Gamma \vdash \Delta$ (resp. of a sequent $\Gamma \vdash \Delta, A[\tau_1/x], A[\tau_2/x]$), either there is a proof of the sequent $A[\tau_1/x], \Gamma \vdash \Delta$, or there is a proof of the sequent $A[\tau_2/x], \Gamma \vdash \Delta$ (resp. either there is a proof of the sequent $\Gamma \vdash \Delta, A[\tau_1/x]$, or there is a proof of the sequent $\Gamma \vdash \Delta, A[\tau_2/x]$). Furthermore, this new proof does not use more contractions than Π does.*

Proof. The proof is by induction on the structure of the proof tree. Lemma 3 is used in the proof. □

We can now prove that in classical logic, the contraction rules are only needed for the quantifier-rules.

Lemma 5 (Contraction elimination) *Every proof in $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\neg}$ can be transformed to a proof in $\mathcal{GK}_c^{\supset,\wedge,\vee,\forall,\exists,\neg}$.*

Proof. The proof is by lexicographic induction on the pair $\langle m, n\rangle$, where m is the number of contractions in the proof tree Π, and n is the size of Π. We use Lemma 4 when the last inference is a contraction, with $\tau_1 = \tau_2 = x$. Since at least one contraction goes away, we can apply the induction hypothesis. □

We now present a cut-free system for intuitionistic logic which does not include any explicit contraction rule and in which the premise of every sequent can be interpreted as a set. Using this system $\mathcal{GK}_i$ due to Kleene (see system $G3a$, page 481, in [11]), we can give a very nice proof of the decidability of provability in intuitionistic propositional logic. The idea behind this system is to systematically keep a copy of the principal formula in the premise(s) of every left-rule. Since Lemma 5 *fails* for intuitionistic logic, such a system is of interest.

Definition 19 *The axioms and inference rules of the system $\mathcal{GK}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ for intuitionistic first-order logic are given below.*

$$A, \Gamma \vdash A$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \quad (\bot\text{: } \textit{right})$$

$$\frac{A \wedge B, A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C} \quad (\wedge\text{: } \textit{left}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad (\wedge\text{: } \textit{right})$$

$$\frac{A \vee B, A, \Gamma \vdash C \quad A \vee B, B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C} \quad (\vee\text{: } \textit{left})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad (\vee\text{: } \textit{right}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \quad (\vee\text{: } \textit{right})$$

$$\frac{A \supset B, \Gamma \vdash A \quad A \supset B, B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} \quad (\supset\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\text{: } \textit{right})$$

$$\frac{\forall x A, A[\tau/x], \Gamma \vdash C}{\forall x A, \Gamma \vdash C} \quad (\forall\text{: } \textit{left}) \qquad \frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A} \quad (\forall\text{: } \textit{right})$$

where in ($\forall$: right), y does not occur free in the conclusion;

$$\frac{\exists x A, A[y/x], \Gamma \vdash C}{\exists x A, \Gamma \vdash C} \quad (\exists\text{: } \textit{left}) \qquad \frac{\Gamma \vdash A[\tau/x]}{\Gamma \vdash \exists x A} \quad (\exists\text{: } \textit{right})$$

where in ($\exists$: left), y does not occur free in the conclusion.

The variable y is called the eigenvariable of the inference.

The following lemma shows that $\mathcal{GK}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ is equivalent to $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$, and also that the premise of every sequent of $\mathcal{GK}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ can be viewed as a set.

Lemma 6 *For every sequent $\Gamma \vdash C$, every proof Π in $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ can be transformed into a proof Π' of $\Gamma \vdash C$ in $\mathcal{GK}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$. Furthermore, a proof Π' can be found such that every formula occurring on the left of any sequent in Π' occurs exactly once. In other words, for every sequent $\Gamma \vdash C$ in Π', the premise Γ can be viewed as a set.*

Proof. The proof is by induction on the structure of Π. The case where the last inference (at the root of the tree) is a contraction follows by induction. Otherwise, the sequent to be proved is either of the form $\Gamma \vdash D$ where Γ is a set, or it is of the form $\Delta, A, A \vdash D$. The first case reduces to the second since Γ can be written as Δ, A, and from a proof of $\Delta, A \vdash D$, we easily obtain a proof of $\Delta, A, A \vdash D$. If the last inference applies to a formula in Δ or D, the induction hypothesis yields the desired result. If the last inference applies to one of the two A's, we apply the induction hypothesis and observe that the rules of $\mathcal{GK}_i$ have been designed to automatically contract the two occurrences of A that would normally be created. For example, if $A = B \wedge C$, the induction hypothesis would yield a proof of $\Delta, B \wedge C, B, C \vdash D$ considered as a set, and the ($\wedge$: *left*)-rule of $\mathcal{GK}_i$ yields $\Delta, B \wedge C \vdash D$ considered as a set. □

As a corollary of Lemma 6 we obtain the fact that provability is decidable for intuitionistic propositional logic. Similarly, Lemma 5 implies that provability is decidable for classical propositional logic.

Theorem 6 *It is decidable whether a proposition is provable in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$. It is decidable whether a proposition is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\bot,cut}$.*

Proof. By the existence of the functions $\mathcal{N}$ and $\mathcal{G}$, there is a proof of a proposition A in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$ iff there is a proof of the sequent $\vdash A$ in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot,cut}$. By the cut elimination theorem (Theorem 5), there is a proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot,cut}$ iff there is a proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$. By Lemma 6, there is a proof in $\mathcal{G}_i^{\supset,\wedge,\vee,\bot}$ iff there is a proof in $\mathcal{GK}_i^{\supset,\wedge,\vee,\bot}$. Call a proof *irredundant* if for every sequent $\Gamma \vdash C$ in this proof, Γ is a set, and no sequent occurs twice on any path. If a proof contains a redundant sequent $\Gamma \vdash C$ occurring at two locations on a path, it is clear that this proof can be shortened by replacing the subproof rooted at the lower (closest to the root) location of the repeating sequent $\Gamma \vdash C$ by the smaller subproof rooted at the higher location of the sequent $\Gamma \vdash C$. Thus, a redundant proof can always be converted to an irredundant proof of the same sequent. Since premises of sequents can be viewed as sets and we are considering cut-free proofs, only subformulae of the formulae occurring in the original sequent to be proved can occur in any proof of that sequent. Therefore, there is a fixed bound

on the size of every irredundant proof of a given sequent. Thus, one simply has to search for an irredundant proof of the given sequent.

By the cut elimination theorem (Theorem 5), there is a proof in $\mathcal{G}_c^{\supset,\wedge,\vee,\bot,cut}$ iff there is a proof in $\mathcal{G}_c^{\supset,\wedge,\vee,\bot}$. By Lemma 5, there is a proof in $\mathcal{G}_c^{\supset,\wedge,\vee,\bot}$ iff there is a proof in $\mathcal{GK}_c^{\supset,\wedge,\vee,\bot}$. To conclude, note that every inference of $\mathcal{GK}_c^{\supset,\wedge,\vee,\bot}$ decreases the total number of connectives in the sequent. Thus, given a sequent, there are only finitely many proofs for it. □

As an exercise, the reader can show that the proposition

$$\big((P \supset Q) \supset P\big) \supset P,$$

known as *Pierce's law*, is not provable in $\mathcal{N}_i^{\supset,\wedge,\vee,\bot}$, but is provable classically in $\mathcal{N}_c^{\supset,\wedge,\vee,\bot}$.

The fact that in any cut-free proof (intuitionistic or classical) of a propositional sequent only subformulae of the formulae occurring in that sequent can occur in the proof is an important property called the *subformula property*. The subformula property is not preserved by the quantifier rules, and this suggests that provability in first-order intuitionistic logic or classical logic is undecidable. This can indeed be shown.

One of the major differences between Gentzen systems for intuitionistic and classical logic presented so far, is that in intuitionistic systems, sequents are restricted to have at most one formula on the righthand side of $\vdash$. This assymetry causes the ($\supset$: *left*) and ($\vee$: *right*) rules of intuitionistic logic to be different from their counterpart in classical logic, and in particular, the intuitionistic rules cause some *loss* of information. For instance, the intuitionistic ($\supset$: *left*)-rule is

$$\frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} \quad (\supset\text{: } \textit{left})$$

whereas its classical version is

$$\frac{\Gamma \vdash A, C \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} \quad (\supset\text{: } \textit{left})$$

Note that C is dropped in the left premise of the intuitionistic version of the rule. Similarly, the intuitionistic ($\vee$: *right*)-rules are

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad (\vee\text{: } \textit{right}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \quad (\vee\text{: } \textit{right})$$

whereas the classical version is

$$\frac{\Gamma \vdash A, B}{\Gamma \vdash A \vee B} \quad (\vee\text{: } \textit{right})$$

Again, either A or B is dropped in the premise of the intuitionistic version of the rule. This loss of information is responsible for the fact that in searching for a proof of a sequent in

$\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$, one cannot stop after having found a deduction tree which is not a proof (*i.e.* a deduction tree in which some leaf is not labeled with an axiom). The rules may have been tried in the wrong order, and it is necessary to make sure that *all* attempts have failed to be sure that a sequent is not provable (in fact, this search should be conducted in the system $\mathcal{GK}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ to ensure termination in the propositional case).

Takeuti [20] has made an interesting observation about intuitionistic cut-free sequent calculi, but before discussing this observation, we shall briefly discuss some recent results of Lincoln, Scedrov, and Shankar [12], about propositional intuitionistic logic based on the connective $\supset$. The system of Definition 19 restricted to propositions built up only from $\supset$ is shown below:

$$A, \Gamma \vdash A$$

$$\frac{A \supset B, \Gamma \vdash A \quad A \supset B, B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} \quad (\supset\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\text{: } \textit{right})$$

This system is contraction-free, but it is not immediately obvious that provability is decidable, since $A \supset B$ is recopied in the premises of the ($\supset$: *left*)-rule. First, it is easy to see that we can require A in an axiom to be atomic, and to see that we can drop $A \supset B$ from the right premise and obtain an equivalent system. The new rule is

$$\frac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} \quad (\supset\text{: } \textit{left})$$

Indeed, if we have a proof of $A \supset B, B, \Gamma \vdash C$, since $B, \Gamma \vdash A \supset B$ is provable, by a cut we obtain that the sequent $B, \Gamma \vdash C$ is provable. Now, the difficulty is to weaken the hypothesis $A \supset B$ in the left premise. What is remarkable is that when A itself is an implication, that is when $A \supset B$ is of the form $(A' \supset B') \supset B$, then $((A' \supset B') \supset B) \supset (B' \supset B)$ is provable, and $B' \supset B$ does indeed work. Also, when A is atomic, then it can be shown that recopying $A \supset B$ is redundant. The new system introduced by Lincoln, Scedrov, and Shankar, is the following:

$$P, \Gamma \vdash P$$

$$\frac{\Gamma \vdash P \quad B, \Gamma \vdash C}{P \supset B, \Gamma \vdash C} \quad (\supset\text{: } \textit{left}) \qquad \frac{B \supset C, \Gamma \vdash A \supset B \quad C, \Gamma \vdash D}{(A \supset B) \supset C, \Gamma \vdash D} \quad (\supset\text{: } \textit{left})$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\text{: } \textit{right})$$

where P is atomic.

The equivalence of this new system and of the previous one is nontrivial. One of the steps involved is "depth-reduction". This means that we can restrict ourselves to propositions which when viewed as trees have depth at most 2 (thus, such a formula is of the form P, $(P \supset Q)$,

$P \supset (Q \supset R)$, or $(P \supset Q) \supset R$, where P, Q, R are atomic). A nice feature of this new system is that it yields easily the decidability of provability. Note that under the multiset ordering, the complexity of the premises of each rule decreases strictly (we consider the multiset of the number of connectives in the formulae occurring in each sequent). For example, $(A \supset B) \supset C$ is replaced by $B \supset C$ and $A \supset B$, both of (strictly) smaller complexity.

We now come back to Takeuti's observation [20]. The crucial fact about intuitionistic systems is not so much the fact that sequents are restricted so that righthand sides have at most one formula, but that the application of the rules ($\supset$: *right*) and ($\forall$: *right*) should be *restricted* so that the righthand side of the conclusion of such a rule consists of a *single* formula (and similarly for ($\neg$: *right*) if $\neg$ is not treated as an abbreviation). The intuitive reason is that the rule ($\supset$: *right*) moves some formula from the lefthand side to the righthand side of a sequent (and similarly for ($\neg$: *right*)), and ($\forall$: *right*) involves a side condition. Now, we can view a classical sequent $\Gamma \vdash B_1, \ldots, B_n$ as the corresponding intuitionistic sequent $\Gamma \vdash B_1 \vee \ldots \vee B_n$. With this in mind, we can show the following result.

Lemma 7 *Let $\mathcal{GT}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$ be the system $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\perp}$ where the application of the rules ($\supset$: right) and ($\forall$: right) is restricted to situations in which the conclusion of the inference is a sequent whose righthand side has a single formula. Then, $\Gamma \vdash B_1, \ldots, B_n$ is provable in $\mathcal{GT}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$ iff $\Gamma \vdash B_1 \vee \ldots \vee B_n$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$.*

Proof. The proof is by induction on the structure of proofs. In the case of an axiom $A, \Gamma \vdash \Delta, A$, letting D be the disjunction of the formulae in Δ, we easily obtain a proof of $A, \Gamma \vdash D \vee A$ in $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$ by applications of ($\vee$: *right*) to the axiom $A, \Gamma \vdash A$. It is also necessary to show that a number of intuitionistic sequents are provable. For example, we need to show that the following sequents are intuitionistically provable:

$$\begin{aligned} A \supset B, (A \vee D) \wedge (B \supset D) &\vdash D, \\ (A \vee D) \wedge (B \vee D) &\vdash (A \wedge B) \vee D, \\ A[t/x] \vee D &\vdash \exists x A \vee D, \\ D \vee \perp &\vdash D \vee A, \\ C \vee C \vee D &\vdash C \vee D, \\ A \vee D, \neg A &\vdash D. \end{aligned}$$

Going from $\mathcal{GT}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$ to $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$, it is much easier to assume that the cut rule can be used in $\mathcal{G}_i$, and then use cut elimination. For example, if the last inference is

$$\frac{\Gamma \vdash A, \Delta \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \quad (\supset\text{: } \textit{left})$$

letting D be the disjunction of the formulae in Δ, by the induction hypothesis, we have proofs in $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\perp}$ of $\Gamma \vdash A \vee D$ and $B, \Gamma \vdash D$. It is obvious that we also have proofs of $A \supset B, \Gamma \vdash A \vee D$ and $A \supset B, \Gamma \vdash B \supset D$, and thus a proof of $A \supset B, \Gamma \vdash (A \vee D) \wedge (B \supset D)$.

Since the sequent $A \supset B, \Gamma, (A \vee D) \wedge (B \supset D) \vdash D$ is provable, using a cut, we obtain that $A \supset B, \Gamma \vdash D$ is provable, as desired. The other cases are similar. □

We can also adapt the system $\mathcal{GK}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ to form a system $\mathcal{GKT}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ having the same property as $\mathcal{GT}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$. In this system, it is also necessary to recopy the principal formula of every right-rule on the righthand side. Such a system can be shown to be complete *w.r.t.* Kripke semantics, and can be used to show the existence of a finite counter-model in the case of a refutable proposition. This system is given in the next definition.

Definition 20 *The axioms and inference rules of the system $\mathcal{GKT}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$ are given below.*

$$A, \Gamma \vdash \Delta, A$$

$$\frac{\Gamma \vdash \Delta, \bot}{\Gamma \vdash \Delta, A} \quad (\bot\text{: } \textit{right})$$

$$\frac{A \wedge B, A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad (\wedge\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A, A \wedge B \quad \Gamma \vdash \Delta, B, A \wedge B}{\Gamma \vdash \Delta, A \wedge B} \quad (\wedge\text{: } \textit{right})$$

$$\frac{A \vee B, A, \Gamma \vdash \Delta \quad A \vee B, B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \quad (\vee\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A, B, A \vee B}{\Gamma \vdash \Delta, A \vee B} \quad (\vee\text{: } \textit{right})$$

$$\frac{A \supset B, \Gamma \vdash A, \Delta \quad A \supset B, B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \quad (\supset\text{: } \textit{left}) \qquad \frac{A, \Gamma \vdash \Delta, B, A \supset B}{\Gamma \vdash \Delta, A \supset B} \quad (\supset\text{: } \textit{right})$$

$$\frac{\forall x\, A, A[\tau/x], \Gamma \vdash \Delta}{\forall x\, A, \Gamma \vdash \Delta} \quad (\forall\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A[y/x], \forall x\, A}{\Gamma \vdash \Delta, \forall x\, A} \quad (\forall\text{: } \textit{right})$$

where in ($\forall$*: right), y does not occur free in the conclusion;*

$$\frac{\exists x\, A, A[y/x], \Gamma \vdash \Delta}{\exists x\, A, \Gamma \vdash \Delta} \quad (\exists\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A[\tau/x], \exists x\, A}{\Gamma \vdash \Delta, \exists x\, A} \quad (\exists\text{: } \textit{right})$$

where in ($\exists$*: left), y does not occur free in the conclusion.*

In the system $\mathcal{GKT}_i^{\supset,\wedge,\vee,\forall,\exists,\bot}$, the application of the rules ($\supset$: *right*) and ($\forall$: *right*) is restricted to situations in which the conclusion of the inference is a sequent whose righthand side has a single formula.

We now consider some equivalent Gentzen systems.

9 The Gentzen Systems $\mathcal{LJ}$ and $\mathcal{LK}$

Axioms of the form $A, \Gamma \vdash \Delta, A$ are very convenient for searching for proofs backwards, but for logical purity, it may be desirable to consider axioms of the form $A \vdash A$. We can redefine axioms to be of this simpler form, but to preserve exactly the same notion of provability, we need to add the following rules of *weakening* (also called *thinning*).

Definition 21 *The rules of weakening (or thinning) are*

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \textit{(weakening: left)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \quad \textit{(weakening: right)}$$

In the case of intuitionistic logic, we require that Δ be empty in (weakening: right).

In view of the previous section, it is easy to see that the quantifier rule

$$\frac{A[\tau/x], \forall x A, \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \quad (\forall\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, \exists x A, A[\tau/x]}{\Gamma \vdash \Delta, \exists x A} \quad (\exists\text{: } \textit{right})$$

of $\mathcal{GK}_c^{\supset,\wedge,\vee,\forall,\exists,\bot}$ are equivalent to the weaker rules

$$\frac{A[\tau/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \quad (\forall\text{: } \textit{left}) \qquad \frac{\Gamma \vdash \Delta, A[\tau/x]}{\Gamma \vdash \Delta, \exists x A} \quad (\exists\text{: } \textit{right})$$

of $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\bot}$, provided that we add (*contrac*: *left*), (*contrac*: *right*), (*weakening*: *left*) and (*weakening*: *right*). Similarly, in order to make the ($\wedge$: *left*) rule and the ($\vee$: *right*) rule analogous to the corresponding introduction rules in natural deduction, we can introduce the rules

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad (\wedge\text{: } \textit{left}) \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad (\wedge\text{: } \textit{left})$$

and

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \quad (\vee\text{: } \textit{right}) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \quad (\vee\text{: } \textit{right})$$

They are equivalent to the old rules provided that we add (*contrac*: *left*), (*contrac*: *right*), (*weakening*: *left*) and (*weakening*: *right*). This leads us to the systems $\mathcal{LJ}$ and $\mathcal{LK}$ defined and studied by Gentzen [3] (except that Gentzen also had an explicit exchange rule, but we assume that we are dealing with multisets).

Definition 22 *The axioms and inference rules of the system $\mathcal{LJ}$ for intuitionistic first-order logic are given below.*

Axioms:

$$A \vdash A$$

Structural Rules:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \textit{(weakening: left)} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A} \quad \textit{(weakening: right)}$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \textit{(contrac: left)}$$

$$\frac{\Gamma \vdash A \quad A, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Theta} \quad \textit{(cut)}$$

Logical Rules:

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad (\wedge\textit{: left}) \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad (\wedge\textit{: left})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad (\wedge\textit{: right})$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \quad (\vee\textit{: left})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad (\vee\textit{: right}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \quad (\vee\textit{: right})$$

$$\frac{\Gamma \vdash A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \quad (\supset\textit{: left}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \quad (\supset\textit{: right})$$

$$\frac{\Gamma \vdash A}{\neg A, \Gamma \vdash} \quad (\neg\textit{: left}) \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \quad (\neg\textit{: right})$$

In the rules above, $A \vee B$, $A \wedge B$, $A \supset B$, and $\neg A$ are called the principal formulae and A, B the side formulae of the inference.

$$\frac{A[\tau/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \quad (\forall\textit{: left}) \qquad \frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A} \quad (\forall\textit{: right})$$

where in ($\forall$: right), y does not occur free in the conclusion;

$$\frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \quad (\exists\textit{: left}) \qquad \frac{\Gamma \vdash A[\tau/x]}{\Gamma \vdash \exists x A} \quad (\exists\textit{: right})$$

where in ($\exists$: left), y does not occur free in the conclusion.

In the above rules, Δ and Θ consist of at most one formula. The variable y is called the eigenvariable of the inference. The condition that the eigenvariable does not occur free in the conclusion of the rule is called the eigenvariable condition. The formula $\forall x A$ (or $\exists x A$) is called the principal formula of the inference, and the formula $A[\tau/x]$ (or $A[y/x]$) the side formula of the inference.

Definition 23 *The axioms and inference rules of the system $\mathcal{LK}$ for classical first-order logic are given below.*

Axioms:

$$A \vdash A$$

Structural Rules:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \textit{(weakening: left)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \quad \textit{(weakening: right)}$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \textit{(contrac: left)} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \quad \textit{(contrac: right)}$$

$$\frac{\Gamma \vdash A, \Delta \quad A, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta} \quad \textit{(cut)}$$

Logical Rules:

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad (\wedge\textit{: left)} \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad (\wedge\textit{: left)}$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \quad (\wedge\textit{: right)}$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \quad (\vee\textit{: left)}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \quad (\vee\textit{: right)} \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \quad (\vee\textit{: right)}$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \quad (\supset\textit{: left)} \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B} \quad (\supset\textit{: right)}$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \quad (\neg\textit{: left)} \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \quad (\neg\textit{: right)}$$

In the rules above, $A \vee B$, $A \wedge B$, $A \supset B$, and $\neg A$ are called the principal formulae and A, B the side formulae of the inference.

$$\frac{A[\tau/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \quad (\forall\textit{: left)} \qquad \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x A} \quad (\forall\textit{: right)}$$

where in ($\forall$: right), y does not occur free in the conclusion;

$$\frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \quad (\exists\textit{: left)} \qquad \frac{\Gamma \vdash \Delta, A[\tau/x]}{\Gamma \vdash \Delta, \exists x A} \quad (\exists\textit{: right)}$$

where in ($\exists$: left), y does not occur free in the conclusion.

The variable y is called the eigenvariable of the inference. The condition that the eigenvariable does not occur free in the conclusion of the rule is called the eigenvariable condition. The formula $\forall x\, A$ (or $\exists x\, A$) is called the principal formula of the inference, and the formula $A[\tau/x]$ (or $A[y/x]$) the side formula of the inference.

One will note that the cut rule

$$\frac{\Gamma \vdash A, \Delta \quad A, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta} \quad (\textit{cut})$$

(with Δ empty in the intuitionistic case and Θ at most one formula) differs from the cut rule

$$\frac{\Gamma \vdash A, \Delta \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \quad (\textit{cut})$$

used in $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\neg,cut}$ or in $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot,cut}$, in that the premises do not require the contexts Γ, Λ to coincide, and the contexts Δ, Θ to coincide. The rules are equivalent using contraction and weakening. Similarly, the other logical rules of $\mathcal{LK}$ (resp. $\mathcal{LJ}$) and $\mathcal{G}_c^{\supset,\wedge,\vee,\forall,\exists,\neg,cut}$ (resp. $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot,cut}$) are equivalent using contraction and weakening.

10 A Proof-Term Calculus for $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot,cut}$

Before we move on to cut elimination in $\mathcal{LK}$ (and $\mathcal{LJ}$), it is worth describing a term calculus corresponding to the sequent calculus $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot,cut}$. A sequent $\Gamma \vdash A$ becomes a judgement $\Gamma^* \triangleright M\colon A$, such that, if $\Gamma = A_1, \ldots, A_n$ then $\Gamma^* = x_1\colon A_1, \ldots, x_n\colon A_n$ is a context in which the x_i are distinct variables and M is a proof term. Since the sequent calculus has rules for introducing formulae on the left of a sequent as well as on the right of a sequent, we will have to create new variables to tag the newly created formulae, and some new term constructors.

Definition 24 *The term calculus associated with $\mathcal{G}_i^{\supset,\wedge,\vee,\forall,\exists,\bot,cut}$ is defined as follows.*

$$\Gamma, x\colon A \triangleright x\colon A$$

$$\frac{x\colon A, y\colon A, \Gamma \triangleright M\colon B}{z\colon A, \Gamma \triangleright \texttt{let } z \texttt{ be } x\colon A @ y\colon A \texttt{ in } M\colon B} \quad (\textit{contrac: left})$$

$$\frac{\Gamma \triangleright N\colon A \quad x\colon A, \Gamma \triangleright M\colon C}{\Gamma \triangleright M[N/x]\colon C} \quad (\textit{cut})$$

$$\frac{\Gamma \triangleright M\colon \bot}{\Gamma \triangleright \triangle_A(M)\colon A} \quad (\textit{weakening: right})$$

$$\frac{x\colon A, y\colon B, \Gamma \triangleright M\colon C}{z\colon A \wedge B, \Gamma \triangleright \mathtt{let}\ z\ \mathtt{be}\ \langle x\colon A, y\colon B\rangle\ \mathtt{in}\ M\colon C}\quad (\wedge\text{: } \textit{left})$$

$$\frac{\Gamma \triangleright M\colon A \quad \Gamma \triangleright N\colon B}{\Gamma \triangleright \langle M, N\rangle\colon A \wedge B}\quad (\wedge\text{: } \textit{right})$$

$$\frac{x\colon A, \Gamma \triangleright M\colon C \quad y\colon B, \Gamma \triangleright N\colon C}{z\colon A \vee B, \Gamma \triangleright \mathtt{case}\ z\ \mathtt{of}\ \mathtt{inl}(x\colon A) \Rightarrow M \mid \mathtt{inr}(y\colon B) \Rightarrow N\colon C}\quad (\vee\text{: } \textit{left})$$

$$\frac{\Gamma \triangleright M\colon A}{\Gamma \triangleright \mathtt{inl}(M)\colon A \vee B}\quad (\vee\text{: } \textit{right}) \qquad \frac{\Gamma \triangleright M\colon B}{\Gamma \triangleright \mathtt{inr}(M)\colon A \vee B}\quad (\vee\text{: } \textit{right})$$

$$\frac{\Gamma \triangleright M\colon A \quad x\colon B, \Gamma \triangleright N\colon C}{z\colon A \supset B, \Gamma \triangleright N[(zM)/x]\colon C}\quad (\supset\text{: } \textit{left}) \qquad \frac{x\colon A, \Gamma \triangleright M\colon B}{\Gamma \triangleright (\lambda x\colon A.\ M)\colon A \supset B}\quad (\supset\text{: } \textit{right})$$

$$\frac{x\colon A[\tau/t], \Gamma \triangleright M\colon C}{z\colon \forall t A, \Gamma \triangleright M[(z\tau)/x]\colon C}\quad (\forall\text{: } \textit{left}) \qquad \frac{\Gamma \triangleright M\colon A[u/t]}{\Gamma \triangleright (\lambda t\colon \iota.\ M)\colon \forall t A}\quad (\forall\text{: } \textit{right})$$

where u does not occur free in Γ or $\forall t A$;

$$\frac{x\colon A[u/t], \Gamma \triangleright M\colon C}{z\colon \exists t A, \Gamma \triangleright \mathtt{select}\ z\ \mathtt{of}\ \mathtt{pair}(t\colon \iota, x\colon A) \Rightarrow M\colon C}\quad (\exists\text{: } \textit{left})$$

where u does not occur free in Γ, $\exists t A$, or C;

$$\frac{\Gamma \triangleright M\colon A[\tau/t]}{\Gamma \triangleright \mathtt{pair}(\tau, M)\colon \exists t A}\quad (\exists\text{: } \textit{right})$$

It is possible to write reduction rules that correspond to cut-elimination steps. For example,

$$\mathtt{let}\ \langle M, N\rangle\ \mathtt{be}\ \langle x\colon A, y\colon B\rangle\ \mathtt{in}\ P \longrightarrow P[M/x, N/y].$$

It is also possible to specify reduction rules imposing a certain strategy, for example, *eager* or *lazy* evaluation. Such reduction strategies have been considered in this setting by Abramsky [1].

The above proof-term assignment has the property that if $\Gamma \triangleright M\colon A$ is derivable and $\Gamma \subseteq \Delta$, then $\Delta \triangleright M\colon A$ is also derivable. This is because the axioms are of the form $\Gamma, x\colon A \triangleright x\colon A$. We can design a term assignment system for an $\mathcal{LJ}$-style system. In such a system, the axioms are of the form

$$x\colon A \triangleright x\colon A$$

and the proof-term assignment for weakening is as follows:

$$\frac{\Gamma \triangleright M\colon B}{z\colon A, \Gamma \triangleright \mathtt{let}\ z\ \mathtt{be}\ \_\ \mathtt{in}\ M\colon B}\quad (\textit{weakening}\text{: } \textit{left})$$

Note that the above proof-term assignment has the property that if $\Gamma \triangleright M\colon A$ is provable and $\Gamma \subseteq \Delta$, then $\Delta \triangleright N\colon A$ is also derivable for some N easily obtainable from M.

If instead of the above ($\wedge$: *left*) rule, we use the two $\mathcal{LJ}$-style rules

$$\frac{A, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C} \quad (\wedge\text{: } \textit{left}) \qquad \frac{B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C} \quad (\wedge\text{: } \textit{left})$$

then we have the following proof-term assignment:

$$\frac{x\text{: } A, \Gamma \rhd M\text{: } C}{z\text{: } A \wedge B, \Gamma \rhd \mathtt{let}\ z\ \mathtt{be}\ \langle x\text{: } A, \_\rangle\ \mathtt{in}\ M\text{: } C} \quad (\wedge\text{: } \textit{left})$$

$$\frac{y\text{: } B, \Gamma \rhd M\text{: } C}{z\text{: } A \wedge B, \Gamma \rhd \mathtt{let}\ z\ \mathtt{be}\ \langle \_, y\text{: } B\rangle\ \mathtt{in}\ M\text{: } C} \quad (\wedge\text{: } \textit{left})$$

It is then natural to write the normalization rules as

$$\mathtt{let}\ \langle M, N\rangle\ \mathtt{be}\ \langle x\text{: } A, \_\rangle\ \mathtt{in}\ P \longrightarrow P[M/x],$$
$$\mathtt{let}\ \langle M, N\rangle\ \mathtt{be}\ \langle \_, y\text{: } B\rangle\ \mathtt{in}\ P \longrightarrow P[N/y].$$

We note that in the second case, the reduction is *lazy*, in the sense that it is unnecessary to normalize N (or M) since it is discarded. In the first case, the reduction is generally *eager* since both M and N will have to be normalized, unless x or y do not appear in P. Such aspects of lazy or eager evaluation become even more obvious in linear logic, as stressed by Abramsky [1].

11 Cut Elimination in $\mathcal{LK}$ (and $\mathcal{LJ}$)

The cut elimination theorem also applies to $\mathcal{LK}$ and $\mathcal{LJ}$. Historically, this is the version of the cut elimination theorem proved by Gentzen [3] (1935). Gentzen's proof was later simplified by Tait [18] and Girard [8] (especially the induction measure). The proof given here combines ideas from Tait and Girard. The induction measure used is due to Tait [18] (the cut-rank), but the explicit transformations are adapted from Girard [8], [4]. We need to define the cut-rank of a formula and the logical depth of a proof.

Definition 25 *The degree $|A|$ of a formula A is the number of logical connectives in A. Let T be an $\mathcal{LK}$-proof. The cut-rank $c(T)$ of T is defined inductively as follows. If T is an axiom, then $c(T) = 0$. If T is not an axiom, the last inference has either one or two premises. In the first case, the premise of that inference is the root of a subtree T_1. In the second case, the left premise is the root of a subtree T_1, and the right premise is the root of a subtree T_2. If the last inference is not a cut, then if it has a single premise, $c(T) = c(T_1)$, else $c(T) = max(c(T_1), c(T_2))$. If the last inference is a cut with cut formula A, then $c(T) = max(\{|A| + 1, c(T_1), c(T_2)\})$. We also define the logical depth of a proof tree T, denoted as $d(T)$, inductively as follows: $d(T) = 0$, when T is an axiom. If the root of T is a single-premise rule, then if the lowest rule is structural, $d(T) = d(T_1)$, else $d(T) = d(T_1) + 1$. If the root of T is a two-premise rule, then $d(T) = max(d(T_1), d(T_2)) + 1$.*

Thus, for an atomic formula, $|A| = 0$. Note that $c(T) = 0$ iff T is cut free, and that if T contains cuts, then $c(T)$ is 1 + the maximum of the degrees of cut formulae in T. We also need the definition of the function $exp(m, n, p)$.

$$exp(m, 0, p) = p;$$
$$exp(m, n + 1, p) = m^{exp(m,n,p)}.$$

This function grows extremely fast in the argument n. Indeed, $exp(m, 1, p) = m^p$, $exp(m, 2, p) = m^{m^p}$, and in general, $exp(m, n, p)$ is an iterated stack of exponentials of height n, topped with a p:

$$exp(m, n, p) = m^{m^{m^{\cdot^{\cdot^{\cdot^{m^p}}}}}} \Big\}n$$

The main idea is to move the cuts "upward", until one of the two premises involved is an axiom. In attempting to design transformations for converting an $\mathcal{LK}$-proof into a cut-free $\mathcal{LK}$-proof, we have to deal with the case in which the cut formula A is contracted in some premise. A transformation to handle this case is given below.

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, A, A}}{\Gamma \vdash \Delta, A} \qquad \dfrac{\pi_2}{A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, A, A} \qquad \dfrac{\pi_2}{A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, A} \qquad \dfrac{\pi_2}{A, \Lambda \vdash \Theta}}{\dfrac{\Gamma, \Lambda, \Lambda \vdash \Delta, \Theta, \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta}}$$

The symmetric rule in which a contraction takes place in the right subtree is not shown. However, there is a problem with this transformation. The problem is that it yields infinite reduction sequences. Consider the following two transformation steps:

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, C, C}}{\Gamma \vdash \Delta, C} \qquad \dfrac{\dfrac{\pi_2}{C, C, \Lambda \vdash \Theta}}{C, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta} \; (cut)$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta, C, C \quad (\pi_1)}{\Gamma, \Lambda \vdash \Delta, \Theta, C} \quad \dfrac{\dfrac{C, C, \Lambda \vdash \Theta \quad (\pi_2)}{C, \Lambda \vdash \Theta}}{\Gamma, \Lambda, \Lambda \vdash \Delta, \Theta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta} \quad \dfrac{C, C, \Lambda \vdash \Theta \quad (\pi_2)}{C, \Lambda \vdash \Theta}}{\Gamma, \Lambda, \Lambda \vdash \Delta, \Theta, \Theta}$$

$$\Rightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta, C, C \quad (\pi_1)}{\Gamma, \Gamma, \Lambda \vdash \Delta, \Delta, \Theta, C, C}}{\Gamma, \Lambda \vdash \Delta, \Theta, C, C}}{\Gamma, \Lambda \vdash \Delta, \Theta, C} \quad \dfrac{\dfrac{\dfrac{C, C, \Lambda \vdash \Theta \quad (\pi_2)}{C, \Gamma, \Lambda \vdash \Delta, \Theta, C} \quad (\pi_1)}{\Gamma, \Gamma, \Lambda \vdash \Delta, \Delta, \Theta, C, C}}{\Gamma, \Lambda \vdash \Delta, \Theta, C, C}}{\Gamma, \Lambda \vdash \Delta, \Theta, C} \quad \dfrac{C, C, \Lambda \vdash \Theta \quad (\pi_2)}{C, \Lambda \vdash \Theta}}{\Gamma, \Lambda, \Lambda \vdash \Delta, \Theta, \Theta}$$

The pattern with contractions on the left and on the right is repeated.

One solution is to consider a more powerful kind of cut rule. In the sequel, the multiset Γ, nA denotes the multiset consisting of all occurrences of $B \neq A$ in Γ and of $m+n$ occurrences of A where m is the number of occurrences of A in Γ.

Definition 26 *(Extended cut rule)*

$$\dfrac{\Gamma \vdash \Delta, mA \qquad nA, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

where $m, n > 0$.

This rule coincides with the standard cut rule when $m = n = 1$, and it is immediately verified that it can be simulated by an application of the standard cut rule and some applications of the contraction rules. Thus, the system $\mathcal{LK}^+$ obtained from $\mathcal{LK}$ by replacing the cut rule by the extended cut rule is equivalent to $\mathcal{LK}$. From now on, we will be working with $\mathcal{LK}^+$. The problem with contraction is then resolved, since we have the following transformation:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, (m-1)A, A, A \quad (\pi_1)}{\Gamma \vdash \Delta, mA} \qquad nA, \Lambda \vdash \Theta \quad (\pi_2)}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Longrightarrow$$

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \Gamma \vdash \Delta, (m+1)A & nA, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

We now prove the main lemma, for which a set of transformations will be needed.

Lemma 8 (Reduction Lemma, Tait, Girard) *Let Π_1 be an $\mathcal{LK}^+$-proof of $\Gamma \vdash \Delta, mA$, and Π_2 an $\mathcal{LK}^+$-proof of $nA, \Lambda \vdash \Theta$, where $m, n > 0$, and assume that $c(\Pi_1), c(\Pi_2) \leq |A|$. An $\mathcal{LK}^+$-proof Π of $\Gamma, \Lambda \vdash \Delta, \Theta$ can be constructed, such that $c(\Pi) \leq |A|$. We also have $d(\Pi) \leq 2(d(\Pi_1) + d(\Pi_2))$, and if the rules for $\supset$ are omitted, then $d(\Pi) \leq d(\Pi_1) + d(\Pi_2)$.*

Proof. It proceeds by induction on $d(\Pi_1) + d(\Pi_2)$, where Π_1 and Π_2 are the immediate subtrees of the proof tree

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma \vdash \Delta, mA & nA, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

There are several (non-mutually exclusive) cases depending on the structure of the immediate subtrees Π_1 and Π_2.

(1) The root of Π_1 and the root of Π_2 is the conclusion of some logical inference having some occurrence of the cut formula A as principal formula. We say that A is *active*.

Every transformation comes in two versions. The first version corresponds to the case of an application of the standard cut rule. The other version, called the "cross-cuts" version, applies when the extended cut rule is involved.

(i) ($\wedge$: *right*) and ($\wedge$: *left*)

$$\dfrac{\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \Gamma \vdash \Delta, B & \Gamma \vdash \Delta, C\end{array}}{\Gamma \vdash \Delta, B \wedge C} \qquad \dfrac{\begin{array}{c}\pi_3\\ B, \Lambda \vdash \Theta\end{array}}{B \wedge C, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Longrightarrow$$

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_3\\ \Gamma \vdash \Delta, B & B, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

By the hypothesis $c(\Pi_1), c(\Pi_2) \leq |A|$, and it is clear that for the new proof Π we have $c(\Pi) \leq |A|$, since $c(\Pi) = max(\{|B| + 1, c(\pi_1), c(\pi_3)\})$, $|B| + 1 \leq |A|$ (since $A = B \wedge C$),

$c(\pi_1) \leq c(\Pi_1)$, $c(\pi_2) \leq c(\Pi_1)$, and $c(\pi_3) \leq c(\Pi_2)$. It is also easy to establish the upper bound on $d(\Pi)$.

Cross-cuts version. Some obvious simplifications apply when either $m = 0$ or $n = 0$, and we only show the main case where $m, n > 0$. Let $A = B \wedge C$.

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA, B} \qquad \overset{\pi_2}{\Gamma \vdash \Delta, mA, C}}{\Gamma \vdash \Delta, (m+1)A} \qquad \dfrac{\overset{\pi_3}{B, nA, \Lambda \vdash \Theta}}{(n+1)A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Let Π_1' be the proof tree obtained by applying the induction hypothesis to

$$\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA, B} \qquad \dfrac{\overset{\pi_3}{B, nA, \Lambda \vdash \Theta}}{(n+1)A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}$$

and Π_2' the proof tree obtained by applying the induction hypothesis to

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA, B} \qquad \overset{\pi_2}{\Gamma \vdash \Delta, mA, C}}{\Gamma \vdash \Delta, (m+1)A} \qquad \overset{\pi_3}{B, nA, \Lambda \vdash \Theta}}{B, \Gamma, \Lambda \vdash \Delta, \Theta}$$

and finally let Π be

$$\dfrac{\dfrac{\overset{\Pi_1'}{\Gamma, \Lambda \vdash \Delta, \Theta, B} \qquad \overset{\Pi_2'}{B, \Gamma, \Lambda \vdash \Delta, \Theta}}{\Gamma, \Gamma, \Lambda, \Lambda \vdash \Delta, \Delta, \Theta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Since $c(\pi_1) \leq c(\Pi_1)$, $c(\pi_2) \leq c(\Pi_1)$, and $c(\pi_3) \leq c(\Pi_2)$, by the induction hypothesis, $c(\Pi_1'), c(\Pi_2') \leq |A|$, and it is clear that for the new proof Π we have $c(\Pi) \leq |A|$, since $c(\Pi) = max(\{|B| + 1, c(\Pi_1'), c(\Pi_2')\})$, and $|B| + 1 \leq |A|$ (since $A = B \wedge C$). It is also easy to establish the upper bound on $d(\Pi)$.

(ii) ($\vee$: *right*) and ($\vee$: *left*)

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, B}}{\Gamma \vdash \Delta, B \vee C} \qquad \dfrac{\overset{\pi_2}{B, \Lambda \vdash \Theta} \qquad \overset{\pi_3}{C, \Lambda \vdash \Theta}}{B \vee C, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Rightarrow$$

$$\dfrac{\dfrac{}{\Gamma \vdash \Delta, B}\,\pi_1 \qquad \dfrac{}{B, \Lambda \vdash \Theta}\,\pi_2}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

By the hypothesis $c(\Pi_1), c(\Pi_2) \leq |A|$, it is clear that for the new proof Π we have $c(\Pi) \leq |A|$, since $c(\Pi) = max(\{|B| + 1, c(\pi_1), c(\pi_2)\})$, $|B| + 1 \leq |A|$ (since $A = B \vee C$), $c(\pi_1) \leq c(\Pi_1)$, $c(\pi_2) \leq c(\Pi_2)$, and $c(\pi_3) \leq c(\Pi_2)$. It is also easy to establish the upper bound on $d(\Pi)$.

Cross-cuts version: Similar to (i) (Some obvious simplifications apply when either $m = 0$ or $n = 0$).

(iii) ($\supset$: *right*) and ($\supset$: *left*)

Left as an exercise.

(iv) ($\neg$: *right*) and ($\neg$: *left*)

$$\dfrac{\dfrac{\dfrac{}{A, \Gamma \vdash \Delta}\,\pi_1}{\Gamma \vdash \Delta, \neg A} \qquad \dfrac{\dfrac{}{\Lambda \vdash \Theta, A}\,\pi_2}{\neg A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Rightarrow$$

$$\dfrac{\dfrac{}{\Lambda \vdash \Theta, A}\,\pi_2 \qquad \dfrac{}{A, \Gamma \vdash \Delta}\,\pi_1}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

By the hypothesis $c(\Pi_1), c(\Pi_2) \leq |\neg A|$, it is clear that for the new proof Π we have $c(\Pi) \leq |\neg A|$, since $c(\Pi) = max(\{|A| + 1, c(\pi_1), c(\pi_2)\})$, $c(\pi_1) \leq c(\Pi_1)$, $c(\pi_2) \leq c(\Pi_2)$. It is also easy to establish the upper bound on $d(\Pi)$.

Cross-cuts version (Some obvious simplifications apply when either $m = 0$ or $n = 0$).

$$\dfrac{\dfrac{\dfrac{}{A, \Gamma \vdash \Delta, m(\neg A)}\,\pi_1}{\Gamma \vdash \Delta, (m+1)(\neg A)} \qquad \dfrac{\dfrac{}{n(\neg A), \Lambda \vdash \Theta, A}\,\pi_2}{(n+1)(\neg A), \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Let Π'_1 be the proof tree obtained by applying the induction hypothesis to

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ A, \Gamma \vdash \Delta, m(\neg A)\end{array}}{\Gamma \vdash \Delta, (m+1)(\neg A)} \qquad \begin{array}{c}\pi_2\\ n(\neg A), \Lambda \vdash \Theta, A\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta, A}$$

and Π_2' the proof tree obtained by applying the induction hypothesis to

$$\dfrac{\begin{array}{c}\pi_1\\ A, \Gamma \vdash \Delta, m(\neg A)\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ n(\neg A), \Lambda \vdash \Theta, A\end{array}}{(n+1)(\neg A), \Lambda \vdash \Theta}}{A, \Gamma, \Lambda \vdash \Delta, \Theta}$$

and finally let Π be

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1'\\ \Gamma, \Lambda \vdash \Delta, \Theta, A\end{array} \qquad \begin{array}{c}\Pi_2'\\ A, \Gamma, \Lambda \vdash \Delta, \Theta\end{array}}{\Gamma, \Gamma, \Lambda, \Lambda \vdash \Delta, \Delta, \Theta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Since $c(\pi_1) \leq c(\Pi_1)$, $c(\pi_2) \leq c(\Pi_2)$, by the induction hypothesis $c(\Pi_1'), c(\Pi_2') \leq |\neg A|$, and it is clear that for the new proof Π we have $c(\Pi) \leq |\neg A|$, since $c(\Pi) = max(\{|A| + 1, c(\Pi_1'), c(\Pi_2')\})$. It is also easy to establish the upper bound on $d(\Pi)$.

(v) ($\forall$: *right*) and ($\forall$: *left*)

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash \Delta, B[y/x]\end{array}}{\Gamma \vdash \Delta, \forall x B} \qquad \dfrac{\begin{array}{c}\pi_2\\ B[t/x], \Lambda \vdash \Theta\end{array}}{\forall x B, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Longrightarrow$$

$$\dfrac{\begin{array}{c}\pi_1[t/x]\\ \Gamma \vdash \Delta, B[t/x]\end{array} \qquad \begin{array}{c}\pi_2\\ B[t/x], \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

In the above, it may be necessary to rename x so that it is distinct from all eigenvariables and distinct from all variables in t.

By the hypothesis $c(\Pi_1), c(\Pi_2) \leq |\forall x B|$, it is clear that for the new proof Π we have $c(\Pi) \leq |\forall x B|$, since $c(\Pi) = max(\{|B[t/x]| + 1, c(\pi_1[t/x]), c(\pi_2)\})$, $c(\pi_1[t/x]) = c(\pi_1)$, $c(\pi_1) \leq c(\Pi_1)$, and $c(\pi_2) \leq c(\Pi_2)$. It is also easy to establish the upper bound on $d(\Pi)$.

Cross-cuts version (Some obvious simplifications apply when either $m = 0$ or $n = 0$).

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta, m(\forall x B), B[y/x]\end{matrix}}{\Gamma \vdash \Delta, (m+1)(\forall x B)} \qquad \dfrac{\begin{matrix}\pi_2\\ B[t/x], n(\forall x B), \Lambda \vdash \Theta\end{matrix}}{(n+1)(\forall x B), \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Let Π'_1 be the proof tree obtained by applying the induction hypothesis to

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta, m(\forall x B), B[y/x]\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ B[t/x], n(\forall x B), \Lambda \vdash \Theta\end{matrix}}{(n+1)(\forall x B), \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B[y/x]}$$

and Π'_2 the proof tree obtained by applying the induction hypothesis to

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta, m(\forall x B), B[y/x]\end{matrix}}{\Gamma \vdash \Delta, (m+1)(\forall x B)} \qquad \begin{matrix}\pi_2\\ B[t/x], n(\forall x B), \Lambda \vdash \Theta\end{matrix}}{B[t/x], \Gamma, \Lambda \vdash \Delta, \Theta}$$

and finally let Π be

$$\dfrac{\dfrac{\begin{matrix}\Pi'_1[t/x]\\ \Gamma, \Lambda \vdash \Delta, \Theta, B[t/x]\end{matrix} \qquad \begin{matrix}\Pi'_2\\ B[t/x], \Gamma, \Lambda \vdash \Delta, \Theta\end{matrix}}{\Gamma, \Gamma, \Lambda, \Lambda \vdash \Delta, \Delta, \Theta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

In the above, it may be necessary to rename x so that it is distinct from all eigenvariables and distinct from all variables in t.

Since $c(\pi_1) \leq c(\Pi_1)$, and $c(\pi_2) \leq c(\Pi_2)$, by the induction hypothesis, $c(\Pi'_1), c(\Pi'_2) \leq |\forall x B|$, and it is clear that for the new proof Π we have $c(\Pi) \leq |\forall x B|$, since $c(\Pi) = max(\{|B[t/x]| + 1, c(\Pi'_1[t/x]), c(\Pi'_2)\})$ and $c(\Pi'_1[t/x]) = c(\Pi'_1)$. It is also easy to establish the upper bound on $d(\Pi)$.

(vi) ($\exists$: *right*) and ($\exists$: *left*)

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta, B[t/x]\end{matrix}}{\Gamma \vdash \Delta, \exists x B} \qquad \dfrac{\begin{matrix}\pi_2\\ B[y/x], \Lambda \vdash \Theta\end{matrix}}{\exists x B, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Rightarrow$$

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, B[t/x]} \qquad \dfrac{\pi_2[t/x]}{B[t/x], \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

In the above, it may be necessary to rename x so that it is distinct from all eigenvariables and distinct from all variables in t.

By the hypothesis, $c(\Pi_1), c(\Pi_2) \leq |\exists x B|$, and it is clear that for the new proof Π we have $c(\Pi) \leq |\exists x B|$, since $c(\Pi) = max(\{|B[t/x]| + 1, c(\pi_1), c(\pi_2[t/x])\})$, $c(\pi_2[t/x]) = c(\pi_2)$, $c(\pi_1) \leq c(\Pi_1)$, and $c(\pi_2) \leq c(\Pi_2)$. It is also easy to establish the upper bound on $d(\Pi)$.

Cross-cuts version (Some obvious simplifications apply when either $m = 0$ or $n = 0$). Similar to (v) and left as an exercise.

(2) Either the root of Π_1 or the root of Π_2 is the conclusion of some logical rule, the cut rule, or some structural rule having some occurrence of a formula $X \neq A$ as principal formula. We say that A is *passive*.

We only show the transformations corresponding to the case where A is passive on the left, the case in which it is passive on the right being symmetric. For this case (where A is passive on the left), we only show the transformation where the last inference applied to the left subtree is a right-rule, the others being symmetric.

(i) ($\vee$: *right*)

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, mA, B}}{\Gamma \vdash \Delta, mA, B \vee C} \qquad \dfrac{\pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B \vee C}$$

$$\Rightarrow$$

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, mA, B} \qquad \dfrac{\pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}}{\Gamma, \Lambda \vdash \Delta, \Theta, B \vee C}$$

Note that $c(\pi_1) \leq c(\Pi_1)$ and $c(\pi_2) \leq c(\Pi_2)$. We conclude by applying the induction hypothesis to the subtree rooted with $\Gamma, \Lambda \vdash \Delta, \Theta, B$. It is also easy to establish the upper bound on $d(\Pi)$.

(ii) ($\wedge$: *right*)

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA, B} \qquad \overset{\pi_2}{\Gamma \vdash \Delta, mA, C}}{\Gamma \vdash \Delta, mA, B \wedge C} \qquad \overset{\pi_3}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B \wedge C}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA, B} \quad \overset{\pi_3}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B} \qquad \dfrac{\overset{\pi_2}{\Gamma \vdash \Delta, mA, C} \quad \overset{\pi_3}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, C}}{\Gamma, \Lambda \vdash \Delta, \Theta, B \wedge C}$$

Note that $c(\pi_1) \leq c(\Pi_1)$, $c(\pi_2) \leq c(\Pi_1)$, and $c(\pi_3) \leq c(\Pi_2)$. We conclude by applying the induction hypothesis to the subtrees rooted with $\Gamma, \Lambda \vdash \Delta, \Theta, B$ and $\Gamma, \Lambda \vdash \Delta, \Theta, C$. It is also easy to establish the upper bound on $d(\Pi)$.

(iii) ($\supset$: *right*)

Left as an exercise.

(iv) ($\neg$: *right*)

$$\dfrac{\dfrac{\overset{\pi_1}{B, \Gamma \vdash \Delta, mA}}{\Gamma \vdash \Delta, mA, \neg B} \qquad \overset{\pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, \neg B}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\overset{\pi_1}{B, \Gamma \vdash \Delta, mA} \qquad \overset{\pi_2}{nA, \Lambda \vdash \Theta}}{B, \Gamma, \Lambda \vdash \Delta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, \neg B}$$

Note that $c(\pi_1) \leq c(\Pi_1)$, and $c(\pi_2) \leq c(\Pi_2)$. We conclude by applying the induction hypothesis to the subtree rooted with $B, \Gamma, \Lambda \vdash \Delta, \Theta$. It is also easy to establish the upper bound on $d(\Pi)$.

(v) ($\forall$: *right*)

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, mA, B[y/x]}}{\Gamma \vdash \Delta, mA, \forall x B} \qquad \overset{\pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, \forall x B}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1[z/x]\\ \Gamma \vdash \Delta, mA, B[z/x]\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta, B[z/x]} \qquad \begin{array}{c}\pi_2\\ nA, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta, \forall x B}$$

In the above, some renaming may be necessary to ensure the eigenvariable condition.

Note that $c(\pi_1[z/x]) = c(\pi_1)$, $c(\pi_1) \leq c(\Pi_1)$, and $c(\pi_2) \leq c(\Pi_2)$. We conclude by applying the induction hypothesis to the subtree rooted with $\Gamma, \Lambda \vdash \Delta, \Theta, B[z/x]$. It is also easy to establish the upper bound on $d(\Pi)$.

(vi) ($\exists$: *right*)

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash \Delta, mA, B[t/x]\end{array}}{\Gamma \vdash \Delta, mA, \exists x B} \qquad \begin{array}{c}\pi_2\\ nA, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash \Delta, mA, B[t/x]\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta, B[t/x]} \qquad \begin{array}{c}\pi_2\\ nA, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta, \exists x B}$$

Note that $c(\pi_1) \leq c(\Pi_1)$, and $c(\pi_2) \leq c(\Pi_2)$. We conclude by applying the induction hypothesis to the subtree rooted with $\Gamma, \Lambda \vdash \Delta, \Theta, B[t/x]$. It is also easy to establish the upper bound on $d(\Pi)$.

(vii) (*cut*)

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma_1 \vdash \Delta_1, m_1 A, pB\end{array} \qquad \begin{array}{c}\pi_2\\ qB, \Lambda_1 \vdash \Theta_1, m_2 A\end{array}}{\Gamma \vdash \Delta, mA} \qquad \begin{array}{c}\pi_3\\ nA, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

where in the above proof π, $m_1 + m_2 = m$, $\Gamma = \Gamma_1, \Lambda_1$, and $\Delta = \Delta_1, \Theta_1$. Since by the hypothesis, $c(\Pi_1), c(\Pi_2) \leq |A|$, and $c(\Pi_1) = max(\{|B| + 1, c(\pi_1), c(\pi_2)\})$, we must have $|B| < |A|$, $c(\pi_1) \leq |A|$, $c(\pi_2) \leq |A|$, and $c(\pi_3) \leq |A|$. Thus in particular, $B \neq A$. We show the transformation in the case where $m_1 > 0$ and $m_2 > 0$, the cases where either $m_1 = 0$ or $m_2 = 0$ being special cases.

Let Π_1' be the result of applying the induction hypothesis to

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_3\\ \Gamma_1 \vdash \Delta_1, pB, m_1 A & nA, \Lambda \vdash \Theta\end{array}}{\Gamma_1, \Lambda \vdash \Delta_1, \Theta, pB}$$

let Π_2' be the result of applying the induction hypothesis to

$$\dfrac{\begin{array}{cc}\pi_2 & \pi_3\\ qB, \Lambda_1 \vdash \Theta_1, m_2 A & nA, \Lambda \vdash \Theta\end{array}}{qB, \Lambda_1, \Lambda \vdash \Theta_1, \Theta}$$

and let Π be the proof

$$\dfrac{\dfrac{\begin{array}{cc}\Pi_1' & \Pi_2'\\ \Gamma_1, \Lambda \vdash \Delta_1, \Theta, pB & qB, \Lambda_1, \Lambda \vdash \Theta_1, \Theta\end{array}}{\Gamma, \Gamma, \Lambda \vdash \Delta, \Delta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Since by the induction hypothesis, $c(\Pi_1'), c(\Pi_2') \leq |A|$, and since $|B| < |A|$, we have $c(\Pi) \leq |A|$. It is also easy to establish the upper bound on $d(\Pi)$.

(viii) (*contrac*: *right*)

$$\dfrac{\dfrac{\pi_1}{\dfrac{\Gamma \vdash \Delta, B, B, mA}{\Gamma \vdash \Delta, B, mA}} \qquad \begin{array}{c}\pi_2\\ nA, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \Gamma \vdash \Delta, B, B, mA & nA, \Lambda \vdash \Theta\end{array}}{\Gamma, \Lambda \vdash \Delta, \Theta, B, B}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}$$

Note that $c(\pi_1) \leq c(\Pi_1)$, and $c(\pi_2) \leq c(\Pi_2)$. We conclude by applying the induction hypothesis to the subtree rooted with $\Gamma, \Lambda \vdash \Delta, \Theta, B, B$. It is also easy to establish the upper bound on $d(\Pi)$.

(ix) (*weakening*: *right*)

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, mA}}{\Gamma \vdash \Delta, mA, B} \qquad \dfrac{\pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta, mA} \qquad \dfrac{\pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, B}$$

Note that $c(\pi_1) \leq c(\Pi_1)$, and $c(\pi_2) \leq c(\Pi_2)$. We conclude by applying the induction hypothesis to the subtrees rooted with $\Gamma, \Lambda \vdash \Delta, \Theta$. It is also easy to establish the upper bound on $d(\Pi)$.

(3) Either Π_1 or Π_2 is an axiom. We consider the case in which the left subtree is an axiom, the other case being symmetric. If $A \notin \Theta$, then

$$\dfrac{A \vdash A \qquad \dfrac{\pi_2}{nA, \Lambda \vdash \Theta}}{A, \Lambda \vdash \Theta}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\pi_2}{nA, \Lambda \vdash \Theta}}{\dfrac{}{A, \Lambda \vdash \Theta}}$$

else

$$\dfrac{A \vdash A \qquad \dfrac{\pi_2}{nA, \Lambda \vdash \Theta, A}}{A, \Lambda \vdash \Theta, A}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{A \vdash A}{}}{A, \Lambda \vdash \Theta, A}$$

Note that $c(\pi_2) \leq c(\Pi_2)$. In the first case, since by hypothesis $c(\Pi_1), c(\Pi_2) \leq |A|$, it is clear that $c(\Pi) \leq |A|$. The second case is obvious.

(4) Either the root of Π_1 or the root of Π_2 is the conclusion of some thinning or contraction resulting in an occurrence of the cut formula A. We consider the case in which this happens in the succedent of the left subtree, the other case being symmetric.

(i) (*weakening*: *right*)

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta\end{matrix}}{\Gamma \vdash \Delta, A} \qquad \begin{matrix}\pi_2\\ A, \Lambda \vdash \Theta\end{matrix}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Longrightarrow$$

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta\end{matrix}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

and when $m, n > 0$,

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta, mA\end{matrix}}{\Gamma \vdash \Delta, (m+1)A} \qquad \begin{matrix}\pi_2\\ nA, \Lambda \vdash \Theta\end{matrix}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\Longrightarrow$$

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta, mA\end{matrix} \qquad \begin{matrix}\pi_2\\ nA, \Lambda \vdash \Theta\end{matrix}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Since by the hypothesis we have $c(\Pi_1), c(\Pi_2) \leq |A|$, it is clear that $c(\Pi) \leq |A|$ in the first case. In the second case, since $c(\pi_1) \leq c(\Pi_1)$ and $c(\pi_2) \leq c(\Pi_2)$, we conclude by applying the induction hypothesis.

(ii) (*contrac*: *right*)

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta, (m-1)A, A, A\end{matrix}}{\Gamma \vdash \Delta, mA} \qquad \begin{matrix}\pi_2\\ nA, \Lambda \vdash \Theta\end{matrix}}{\Gamma, \Lambda \vdash \Delta, \Theta}}{}$$

$$\Longrightarrow$$

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash \Delta, (m+1)A\end{matrix}}{} \qquad \begin{matrix}\pi_2\\ nA, \Lambda \vdash \Theta\end{matrix}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Since by the hypothesis we have $c(\Pi_1), c(\Pi_2) \leq |A|$, and we have $c(\pi_1) \leq c(\Pi_1)$ and $c(\pi_2) \leq c(\Pi_2)$, we conclude by applying the induction hypothesis.

The symmetric rule in which a contraction takes place in the right subtree is not shown. □

We can now prove the following major result (essentially due to Tait [18], 1968), showing not only that every proof can be transformed into a cut-free proof, but also giving an upper bound on the size of the resulting cut-free proof.

Theorem 7 *Let T be a proof with cut-rank $c(T)$ of a sequent $\Gamma \vdash \Delta$. A cut-free proof T^* for $\Gamma \vdash \Delta$ can be constructed such that $d(T^*) \leq exp(4, c(T), d(T))$.*

Proof. We prove the following claim by induction on the depth of proof trees.

Claim: Let T be a proof with cut-rank $c(T)$ for a sequent $\Gamma \vdash \Delta$. If $c(T) > 0$ then we can construct a proof T' for $\Gamma \vdash \Delta$ such that

$$c(T') < c(T) \quad \text{and} \quad d(T') \leq 4^{d(T)}.$$

Proof of Claim: If either the last inference of T is not a cut, or it is a cut and $c(T) > |A| + 1$, we apply the induction hypothesis to the immediate subtrees T_1 or T_2 (or T_1) of T. We are left with the case in which the last inference is a cut and $c(T) = |A| + 1$. The proof is of the form

$$\dfrac{\begin{array}{cc} T_1 & T_2 \\ \Gamma \vdash \Delta, mA & nA, \Gamma \vdash \Delta \end{array}}{\Gamma \vdash \Delta}$$

By the induction hypothesis, we can construct a proof T_1' for $\Gamma \vdash \Delta, mA$ and a proof T_2' for $nA, \Gamma \vdash \Delta$, such that $c(T_i') \leq |A|$ and $d(T_i') \leq 4^{d(T_i)}$, for $i = 1, 2$. Applying the reduction lemma (Lemma 8), we obtain a proof T' such that, $c(T') \leq |A|$ and $d(T') \leq 2(d(T_1') + d(T_2'))$. But

$$2(d(T_1') + d(T_2')) \leq 2(4^{d(T_1)} + 4^{d(T_2)}) \leq 4^{max(d(T_1),d(T_2))+1} = 4^{d(T)}.$$

□

The proof of Theorem 7 follows by induction on $c(T)$, and by the definition of $exp(4, m, n)$. □

It is easily verified that the above argument also goes through for the system $\mathcal{LJ}$. Thus, we obtain Gentzen's original cut elimination theorem.

Theorem 8 (Cut Elimination Theorem, Gentzen (1935)) *There is an algorithm which, given any proof Π in $\mathcal{LK}$ produces a cut-free proof Π' in $\mathcal{LK}$. There is an algorithm which, given any proof Π in $\mathcal{LJ}$ produces a cut-free proof Π' in $\mathcal{LJ}$.*

A few more remarks about the role of contraction and weakening will be useful before moving on to linear logic. We already noticed with the cut rule that contexts (the Γ, Δ occurring in the premise(s) of inference rules) can be treated in two different ways: (1) either they are merged (which implies that they are identical), or (2) they are concatenated.

In order to search for proof backwards, it is more convenient to treat contexts in mode (1), but this hides some subtleties. For example, the ($\wedge$: *right*) rule can be written either as

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

where the contexts are merged, or as

$$\frac{\Gamma \vdash \Delta, A \quad \Lambda \vdash \Theta, B}{\Gamma, \Lambda \vdash \Delta, \Theta, A \wedge B}$$

where the contexts are just concatenated but not merged. Following Girard, let's call the first version *additive*, and the second version *multiplicative*. Under contraction and weakening, the two versions are equivalent: the first rule can be simulated by the second rule using contractions:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma, \Gamma \vdash \Delta, \Delta, A \wedge B}}{\Gamma \vdash \Delta, A \wedge B}$$

and the second rule can be simulated by the first rule using weakenings:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A}{\Gamma, \Lambda \vdash \Delta, \Theta, A} \qquad \dfrac{\Lambda \vdash \Theta, B}{\Gamma, \Lambda \vdash \Delta, \Theta, B}}{\Gamma, \Lambda \vdash \Delta, \Theta, A \wedge B}$$

Similarly, the ($\wedge$: *left*) rules can be written either as

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

or as

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

Again, let's call the first version *additive*, and the second version *multiplicative*. These versions are equivalent under contraction and weakening. The first version can be simulated by the second rule using weakening:

$$\dfrac{\dfrac{A, \Gamma \vdash \Delta}{A, B, \Gamma \vdash \Delta}}{A \wedge B, \Gamma \vdash \Delta} \quad (\textit{weakening}\text{: } \textit{left})$$

and the second version can be simulated by the first rule and contraction:

$$\dfrac{\dfrac{\dfrac{A, B, \Gamma \vdash \Delta}{A \wedge B, B, \Gamma \vdash \Delta}}{A \wedge B, A \wedge B, \Gamma \vdash \Delta}}{A \wedge B, \Gamma \vdash \Delta} \quad (\textit{contrac}\text{: } \textit{left})$$

If we take away contraction and weakening, the additive and multiplicative versions are no longer equivalent. This suggests, and this path was followed by Girard, to split the connectives $\wedge$ and $\vee$ into two versions: the *multiplicative* version of $\wedge$ and $\vee$, denoted as $\otimes$ and $\wp$, and the *additive* version of $\wedge$ and $\vee$, denoted as & and $\oplus$. In *linear logic*, due to Girard [7], the connectives $\wedge$ and $\vee$ are split into multiplicative and additive versions, contraction and weakening are dropped, negation denoted $A^{\perp}$ is involutive, and in order to regain the loss of expressiveness due to the absence of contraction and weakening, some new connectives (the exponentials ! and ?) are introduced. The main role of these connectives is to have better control over contraction and weakening. Thus, at the heart of linear logic lies the notion that resources are taken into account.

12 Reductions of Classical to Intuitionistic Logic

Although there exist formulae that are provable classically but not intuitionistically, there are several ways of embedding classical logic into intuitionistic logic. More specifically, there are functions * from formulae to formulae such that for every formula A, its translation A^* is equivalent to A classically, and A is provable classically iff A^* is provable intuitionistically. Stronger results can be obtained in the propositional case. Since $\neg\neg A \supset A$ is provable classically but not intuitionistically, whereas $A \supset \neg\neg A$ is provable both classically and intuitionistically, we can expect that double-negation will play a crucial role, and this is indeed the case. One of the crucial properties is that triple negation is equivalent to a single negation. This is easily shown as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \neg A \vdash}}{A \vdash \neg\neg A}}{A, \neg\neg\neg A \vdash}}{\neg\neg\neg A \vdash \neg A}$$

Since we also have the following proof

$$\dfrac{\dfrac{\neg A \vdash \neg A}{\neg A, \neg\neg A \vdash}}{\neg A \vdash \neg\neg\neg A}$$

it is clear that $\neg\neg\neg A \equiv \neg A$ is provable intuitionistically.

The possibility of embedding classical logic into intuitionistic logic is due to four crucial facts:

(1) $\neg\neg\neg A \equiv \neg A$ is provable intuitionistically;

(2) If a formula A is provable classically without using the ($\forall$: *right*)-rule, then $\neg\neg A$ is provable intuitionistically;

(3) For a class of formulae for which $\neg\neg A \vdash A$ is provable intuitionistically, (2) holds unrestricted. This means that if a formula A in this class is provable classically then $\neg\neg A$ is provable intuitionistically;

(4) For every formula A built only from $\supset$, $\wedge$, $\neg$ and $\forall$, if $\Delta = \neg\neg P_1 \supset P_1, \ldots, \neg\neg P_k \supset P_k$ where $P_1, \ldots, P_k$ are all the atoms occurring in A, then $\Delta, \neg\neg A \vdash A$ is provable intuitionistically.

The "trick" of the double-negation translation (often attributed to Gödel (1933), although it was introduced independently by Kolmogorov (1925) and Gentzen (1933)) is that if we consider a formula A only built from $\supset$, $\wedge$, $\neg$, $\forall$, and replace every atomic subformula P by $\neg\neg P$ obtaining $A^\dagger$, we get a subclass of formulae for which (4) holds without the Δ, and thus (3) also holds. For this class, A is provable classically iff $A^\dagger$ is provable intuitionistically.

Our first result will concern propositions. Given $\Gamma = A_1, \ldots, A_m$, we let $\neg\neg\Gamma = \neg\neg A_1, \ldots, \neg\neg A_m$.

Lemma 9 *Given a sequent $\Gamma \vdash B_1, \ldots, B_n$ of propositions, if $\Gamma \vdash B_1, \ldots, B_n$ is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg}$, then $\neg\neg\Gamma \vdash \neg(\neg B_1 \wedge \ldots \wedge \neg B_n)$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$.*

Proof. We proceed by induction on proofs. In fact, it is easier to work in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,cut}$ and use cut elimination. It is necessary to prove that a number of propositions are provable intuitionistically. First, observe that if $A_1, \ldots, A_m \vdash B$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,cut}$, then $\neg\neg A_1, \ldots, \neg\neg A_m \vdash \neg\neg B$ is also provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,cut}$. The following sequents are provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,cut}$:

$$\neg\neg A \vdash \neg(\neg A \wedge D),$$

$$\begin{aligned}
\neg(\neg A \wedge D), \neg(\neg B \wedge D) &\vdash \neg(\neg(A \wedge B) \wedge D),\\
\neg\neg A, \neg\neg B &\vdash \neg\neg(A \wedge B),\\
\neg(\neg A \wedge \neg B \wedge D) &\vdash \neg(\neg(A \vee B) \wedge D),\\
\neg\neg A \vee \neg\neg B &\vdash \neg\neg(A \vee B),\\
(\neg\neg A \supset \neg(\neg B \wedge D)) &\vdash \neg(\neg(A \supset B) \wedge D),\\
\neg\neg(A \supset B), \neg(\neg A \wedge D), (\neg\neg B \supset \neg D) &\vdash \neg D,\\
(\neg\neg A \supset \neg D) &\vdash \neg(\neg\neg A \wedge D),\\
\neg(\neg A \wedge D), \neg\neg\neg A &\vdash \neg D,\\
\neg(\neg A \wedge \neg A \wedge D) &\vdash \neg(\neg A \wedge D).
\end{aligned}$$

We now consider the axioms and each inference rule. Given $\Delta = B_1, \ldots, B_n$, we let $D = \neg B_1 \wedge \ldots \wedge \neg B_n$. This way, observe that $\neg(\neg A \wedge \neg B_1 \wedge \ldots \wedge \neg B_n) = \neg(\neg A \wedge D)$. An axiom $\Gamma, A \vdash A, \Delta$ becomes $\neg\neg\Gamma, \neg\neg A \vdash \neg(\neg A \wedge D)$, which is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,cut}$ since $\neg\neg A \vdash \neg(\neg A \wedge D)$ is. Let us also consider the case of the ($\supset$: *right*)-rule, leaving the others as exercises.

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \supset B, \Delta}$$

By the induction hypothesis, $\neg\neg\Gamma, \neg\neg A \vdash \neg(\neg B \wedge D)$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,cut}$, and so is

$$\neg\neg\Gamma \vdash (\neg\neg A \supset \neg(\neg B \wedge D)).$$

Since

$$(\neg\neg A \supset \neg(\neg B \wedge D)) \vdash \neg(\neg(A \supset B) \wedge D)$$

is also provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,cut}$, by a cut, we obtain that

$$\neg\neg\Gamma \vdash \neg(\neg(A \supset B) \wedge D)$$

is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,cut}$, as desired. □

Since $\neg\neg\neg A \equiv \neg A$ is provable intuitionistically, we obtain the following lemma known as Glivenko's Lemma.

Lemma 10 (Glivenko, 1929) *Given a sequent $\neg\Gamma, \Delta \vdash \neg B_1, \ldots, \neg B_n$ made of propositions, if $\neg\Gamma, \Delta \vdash \neg B_1, \ldots, \neg B_n$ is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg}$, then $\neg\Gamma, \neg\neg\Delta \vdash \neg(B_1 \wedge \ldots \wedge B_n)$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$. In particular, if $\neg\Gamma \vdash \neg B$ is a propositional sequent provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg}$, then it is also provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$.*

Proof. By Lemma 9, using the fact that $\neg\neg\neg A \equiv \neg A$ is provable intuitionistically, and that the sequent

$$\neg(\neg\neg B_1 \wedge \ldots \wedge \neg\neg B_n) \vdash \neg(B_1 \wedge \ldots \wedge B_n)$$

is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$. □

As a consequence of Lemma 9, if a proposition A is provable classically, then $\neg\neg A$ is provable intuitionistically, and as a consequence of Lemma 10, if a proposition $\neg A$ is provable classically, then it is also provable intuitionistically. It should be noted that Lemma 9 *fails* for quantified formulae. For example, $\forall x(P(x) \vee \neg P(x))$ is provable classically, but we can show that $\neg\neg\forall x(P(x) \vee \neg P(x))$ is not provable intuitionistically, for instance using the system of Lemma 6. Similarly, $\forall x \neg\neg P(x) \supset \neg\neg\forall x\, P(x)$ is provable classically, but it is not provable intuitionistically, and neither is $\neg\neg(\forall x \neg\neg P(x) \supset \neg\neg\forall x\, P(x))$. As observed by Gödel, Lemma 10 has the following remarkable corollary.

Lemma 11 (Gödel, 1933) *For every proposition A built only from $\wedge$ and $\neg$, if A is provable classically, then A is also provable intuitionistically.*

Proof. By induction on A. If $A = \neg B$, then this follows by Glivenko's Lemma. Otherwise, it must be possible to write $A = B_1 \wedge \ldots \wedge B_n$ where each B_i is not a conjunct and where each B_i is provable classically. Thus, each B_i must be of the form $\neg C_i$, since if B_i is an atom it is not provable. Again, each B_i is provable intuitionistically by Glivenko's Lemma, and thus so is A. □

Lemma 9 indicates that double-negation plays an important role in linking classical logic to intuitionistic logic. The following lemma shows that double-negation distributes over the connectives $\wedge$ and $\supset$.

Lemma 12 *The following formulae are provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$:*

$$\neg\neg(A \wedge B) \equiv \neg\neg A \wedge \neg\neg B,$$
$$\neg\neg(A \supset B) \equiv \neg\neg A \supset \neg\neg B.$$

Proof. We give a proof for

$$\neg\neg(A \supset B) \vdash \neg\neg A \supset \neg\neg B,$$

leaving the others as exercises.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, \neg B \vdash A \qquad \dfrac{A, B \vdash B}{A, B, \neg B \vdash}}{A, A \supset B, \neg B \vdash}}{A, \neg B \vdash \neg(A \supset B)}}{A, \neg\neg(A \supset B), \neg B \vdash}}{\neg\neg(A \supset B), \neg B \vdash \neg A}}{\neg\neg(A \supset B), \neg\neg A, \neg B \vdash}}{\neg\neg(A \supset B), \neg\neg A \vdash \neg\neg B}}{\neg\neg(A \supset B) \vdash \neg\neg A \supset \neg\neg B}$$

□

Lemma 12 *fails* for disjunctions. For example,

$$\neg\neg(P \vee \neg P) \vdash (\neg\neg P \vee \neg\neg\neg P)$$

is *not* provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$, since $\neg\neg(P \vee \neg P)$ is provable but $(\neg\neg P \vee \neg\neg\neg P)$ is *not* provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg}$ (this is easily shown using the system $\mathcal{GK}_i$). Lemma 12 also *fails* for the quantifiers. For example, using the system of Lemma 6, we can show that $\forall x \neg\neg P(x) \supset \neg\neg\forall x P(x)$ and $\neg\neg\exists x P(x) \supset \exists x \neg\neg P(x)$ are not provable intuitionistically.

Even though Lemma 9 fails in general, in particular for universal formulae, Kleene has made the remarkable observation that the culprit is precisely the ($\forall$: *right*)-rule [11] (see Theorem 59, page 492). Indeed, the lemma still holds for arbitrary sequents $\Gamma \vdash B_1, \ldots, B_n$, provided that their proofs in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$ do *not* use the rule ($\forall$: *right*).

Lemma 13 *Given a first-order sequent $\Gamma \vdash B_1, \ldots, B_n$, if $\Gamma \vdash B_1, \ldots, B_n$ is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$ without using the rule ($\forall$: right), then $\neg\neg\Gamma \vdash \neg(\neg B_1 \wedge \ldots \wedge \neg B_n)$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$.*

Proof. As in the proof of Lemma 9, we proceed by induction on proofs. It is necessary to prove that the following sequents are provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$:

$$\begin{aligned}
\neg(\neg A[t/x] \wedge D) &\vdash \neg(\neg\exists x A \wedge D),\\
\forall x(\neg\neg A \supset \neg D), \neg\neg\exists x A &\vdash \neg D,\\
(\neg\neg A[t/x] \supset \neg D), \neg\neg\forall x A &\vdash \neg D.
\end{aligned}$$

where x does not occur in D in the second sequent. Proofs for the above sequents follow:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A[t/x], D \vdash A[t/x]}{A[t/x], D \vdash \exists x A}}{A[t/x], \neg\exists x A, D \vdash}}{\neg\exists x A, D \vdash \neg A[t/x]} \qquad \neg\exists x A, D \vdash D}{\dfrac{\dfrac{\dfrac{\neg\exists x A, D \vdash \neg A[t/x] \wedge D}{\neg(\neg A[t/x] \wedge D), \neg\exists x A, D \vdash}}{\neg(\neg A[t/x] \wedge D), \neg\exists x A \wedge D \vdash}}{\neg(\neg A[t/x] \wedge D) \vdash \neg(\neg\exists x A \wedge D)}}$$

$$\dfrac{\dfrac{\dfrac{D, A[y/x] \vdash A[y/x]}{D, A[y/x], \neg A[y/x] \vdash}}{D, A[y/x] \vdash \neg\neg A[y/x]} \qquad \dfrac{D, A[y/x] \vdash D}{\neg D, D, A[y/x] \vdash}}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\neg A[y/x] \supset \neg D, D, A[y/x] \vdash}{\forall x(\neg\neg A \supset \neg D), D, A[y/x] \vdash}}{\forall x(\neg\neg A \supset \neg D), D, \exists x A \vdash}}{\forall x(\neg\neg A \supset \neg D), D \vdash \neg\exists x A}}{\forall x(\neg\neg A \supset \neg D), \neg\neg\exists x A, D \vdash}}{\forall x(\neg\neg A \supset \neg D), \neg\neg\exists x A \vdash \neg D}}$$

where x does not occur in D, and y is a new variable.

$$\dfrac{\dfrac{\dfrac{D, A[t/x] \vdash A[t/x]}{D, A[t/x], \neg A[t/x] \vdash}}{D, A[t/x] \vdash \neg\neg A[t/x]} \qquad \dfrac{D, A[t/x] \vdash D}{\neg D, D, A[t/x] \vdash}}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\neg\neg A[t/x] \supset \neg D), D, A[t/x] \vdash}{(\neg\neg A[t/x] \supset \neg D), D, \forall x A \vdash}}{(\neg\neg A[t/x] \supset \neg D), D \vdash \neg\forall x A}}{(\neg\neg A[t/x] \supset \neg D), \neg\neg\forall x A, D \vdash}}{(\neg\neg A[t/x] \supset \neg D), \neg\neg\forall x A \vdash \neg D}}$$

We now have to consider the cases where the last inference is one of ($\forall$: *left*), ($\exists$: *left*), or ($\exists$: *right*). We treat the case of the rule ($\exists$: *right*), leaving the others as exercises.

$$\dfrac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}$$

Given $\Delta = B_1, \ldots, B_n$, we let $D = \neg B_1 \wedge \ldots \wedge \neg B_n$. By the induction hypothesis, $\neg\neg\Gamma \vdash \neg(\neg A[t/x] \wedge D)$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$. On the other hand, since the sequent

$$\neg(\neg A[t/x] \wedge D) \vdash \neg(\neg\exists x A \wedge D)$$

is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$, using a cut, we obtain that the sequent

$$\neg\neg\Gamma \vdash \neg(\neg\exists x A \wedge D)$$

is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$, as desired. □

Technically, the problem with Lemma 13, is that the sequent

$$\forall x \neg(\neg A \wedge D) \vdash \neg(\neg \forall x A \wedge D)$$

(where x does not occur in D) is not provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$. In order to see where the problem really lies, we attempt to construct a proof of this sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\neg A[y/x] \wedge D), D \vdash A[y/x]}{\forall x \neg(\neg A \wedge D), D \vdash A[y/x]}}{\forall x \neg(\neg A \wedge D), D \vdash \forall x A}}{\forall x \neg(\neg A \wedge D), \neg \forall x A, D \vdash}}{\forall x \neg(\neg A \wedge D), \neg \forall x A \wedge D \vdash}}{\forall x \neg(\neg A \wedge D) \vdash \neg(\neg \forall x A \wedge D)}$$

where x does not occur in D, and y is a new variable. The problem is that we cannot apply the ($\forall$: *left*)-rule before $\neg \forall x A$ has been transferred to the righthand side of the sequent (as $\forall x A$) and before the ($\forall$: *right*)-rule has been applied to $\forall x A$, since this would violate the eigenvariable condition. Unfortunately, we are stuck with the sequent $\neg(\neg A[y/x] \wedge D), D \vdash A[y/x]$ which is unprovable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$. However, note that the sequent $\neg(\neg A[y/x] \wedge D), D \vdash \neg\neg A[y/x]$ in which $A[y/x]$ has been replaced with $\neg\neg A[y/x]$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$:

$$\dfrac{\dfrac{\dfrac{D, \neg A[y/x] \vdash \neg A[y/x] \qquad D, \neg A[y/x] \vdash D}{D, \neg A[y/x] \vdash \neg A[y/x] \wedge D}}{\neg(\neg A[y/x] \wedge D), D, \neg A[y/x] \vdash}}{\neg(\neg A[y/x] \wedge D), D \vdash \neg\neg A[y/x]}$$

Thus, if the sequent $\neg\neg A \vdash A$ was provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$, the sequent

$$\forall x \neg(\neg A \wedge D) \vdash \neg(\neg \forall x A \wedge D)$$

would also be provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$. It is therefore important to identify a subclass of first-order formulae for which $\neg\neg A \vdash A$ is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$, since for such a class, Lemma 13 holds without restrictions. The following lemma showing the importance of the axiom $\neg\neg P \vdash P$ where P is atomic, leads us to such a class of formulae. It is at the heart of the many so-called "double-negation translations".

Lemma 14 *For every formula A built only from $\supset$, $\wedge$, $\neg$, $\forall$, the sequent $\neg\neg A \vdash A$ is provable in the system $\mathcal{G}_{i+}^{\supset,\wedge,\neg,\forall}$ obtained from $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$ by adding all sequents of the form $\neg\neg P \vdash P$ where P is atomic as axioms. Equivalently, if $\Delta = \neg\neg P_1 \supset P_1, \ldots, \neg\neg P_k \supset P_k$ where $P_1, \ldots, P_k$ are all the atoms occurring in A, then $\Delta, \neg\neg A \vdash A$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$.*

Proof. It proceeds by induction on the structure of A. If A is an atom P, this is obvious since $\neg\neg P \vdash P$ is an axiom. If $A = B \wedge C$, by the induction hypothesis, both $\neg\neg B \vdash B$ and $\neg\neg C \vdash C$ are provable in $\mathcal{G}_{i+}^{\supset,\wedge,\neg,\forall}$, and so is $\neg\neg C \wedge \neg\neg C \vdash B \wedge C$. We just have to prove that $\neg\neg(A \wedge B) \vdash \neg\neg A \wedge \neg\neg B$ is provable in $\mathcal{G}_{i}^{\supset,\wedge,\neg,\forall}$, which is easily done. If $A = \neg B$, since we have shown that $\neg\neg\neg B \vdash \neg B$ is provable in $\mathcal{G}_{i}^{\supset,\wedge,\neg,\forall}$, so is $\neg\neg A \vdash A$. If $A = B \supset C$, then by the induction hypothesis, $\neg\neg C \vdash C$ is provable in $\mathcal{G}_{i+}^{\supset,\wedge,\neg,\forall}$ (and so is $\neg\neg B \vdash B$, but we won't need it). Observe that the sequent $\neg\neg C \supset C, \neg\neg(B \supset C) \vdash B \supset C$ is provable in $\mathcal{G}_{i}^{\supset,\wedge,\neg,\forall}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B \qquad B, C \vdash C}{B, (B \supset C) \vdash C}}{B, \neg C, (B \supset C) \vdash}}{B, \neg C \vdash \neg(B \supset C)}}{\neg\neg(B \supset C), B, \neg C \vdash}}{\neg\neg(B \supset C), B \vdash \neg\neg C} \qquad \neg\neg(B \supset C), B, C \vdash C}{\neg\neg C \supset C, \neg\neg(B \supset C), B \vdash C}}{\neg\neg C \supset C, \neg\neg(B \supset C) \vdash B \supset C}$$

Using the fact that $\neg\neg C \vdash C$ is provable in $\mathcal{G}_{i+}^{\supset,\wedge,\neg,\forall}$ and a suitable cut, $\neg\neg(B \supset C) \vdash B \supset C$ is provable in $\mathcal{G}_{i+}^{\supset,\wedge,\neg,\forall}$. If $A = \forall x B$, we can show easily that $\neg\neg\forall x B \vdash \neg\neg B[t/x]$ is provable in $\mathcal{G}_{i}^{\supset,\wedge,\neg,\forall}$. Since by the induction hypothesis, $\neg\neg B \vdash B$ is provable in $\mathcal{G}_{i+}^{\supset,\wedge,\neg,\forall}$, for any new variable y, $\neg\neg B[y/x] \vdash B[y/x]$ is also provable in $\mathcal{G}_{i+}^{\supset,\wedge,\neg,\forall}$, and thus by choosing $t = y$, the sequent $\neg\neg\forall x B \vdash B[y/x]$ is provable where y is new, so that $\neg\neg\forall x B \vdash \forall x B$ is provable in $\mathcal{G}_{i+}^{\supset,\wedge,\neg,\forall}$. □

In order to appreciate the value of Lemma 14, the reader should find a direct proof of $\neg\neg(\neg\neg P \supset P)$ in $\mathcal{G}_{i}^{\supset,\wedge,\neg,\forall}$. Unfortunately, Lemma 14 *fails* for disjunctions and existential quantifiers. For example,

$$\neg\neg P \supset P \vdash \neg\neg(P \vee \neg P) \supset (P \vee \neg P)$$

is not provable in $\mathcal{G}_{i}^{\supset,\wedge,\vee,\neg,\forall,\exists}$. This can be shown as follows. Since $P \vee \neg P$ is provable in $\mathcal{G}_{c}^{\supset,\wedge,\vee,\neg,\forall,\exists}$, by Lemma 9, $\neg\neg(P \vee \neg P)$ is provable in $\mathcal{G}_{i}^{\supset,\wedge,\vee,\neg,\forall,\exists}$. Thus, $\neg\neg P \supset P \vdash (P \vee \neg P)$ would be provable in $\mathcal{G}_{i}^{\supset,\wedge,\vee,\neg,\forall,\exists}$, but we can show using the system of Lemma 6 that this is not so.

The sequent

$$\neg\neg P \supset P \vdash \neg\neg(\neg\neg\exists x P(x) \supset \exists x \neg\neg P(x)) \supset (\neg\neg\exists x P(x) \supset \exists x \neg\neg P(x))$$

is also not provable in $\mathcal{G}_{i}^{\supset,\wedge,\vee,\neg,\forall,\exists}$. This is because $(\neg\neg\exists x P(x) \supset \exists x \neg\neg P(x))$ is provable in $\mathcal{G}_{c}^{\supset,\wedge,\vee,\neg,\forall,\exists}$ without using the ($\forall$: *right*)-rule, and so, by Lemma 13, $\neg\neg(\neg\neg\exists x P(x) \supset$

$\exists x \neg\neg P(x)$) is provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$. Then,

$$\neg\neg P \supset P \vdash (\neg\neg\exists x\, P(x) \supset \exists x \neg\neg P(x))$$

would be provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$, but we can show using the system of Lemma 6 that this is not so.

Since the sequent $A \vee \neg A \vdash \neg\neg A \supset A$ is easily shown to be provable in $\mathcal{G}_i^{\supset,\wedge,\vee,\neg,\forall,\exists}$, Lemma 14 also holds with the axioms $\vdash P \vee \neg P$ substituted for $\neg\neg P \vdash P$ (for all atoms P). In fact, with such axioms, we can even show that Lemma 14 holds for disjunctions (but not for existential formulae).

In view of Lemma 14 we can define the following function † on formulae built from $\supset, \wedge, \neg, \forall$:

$$\begin{aligned} A^\dagger &= \neg\neg A, \quad \text{if } A \text{ is atomic,} \\ (\neg A)^\dagger &= \neg A^\dagger, \\ (A * B)^\dagger &= (A^\dagger * B^\dagger), \quad \text{if } * \in \{\supset, \wedge\}, \\ (\forall x\, A)^\dagger &= \forall x\, A^\dagger. \end{aligned}$$

Given a formula built only from $\supset, \wedge, \neg, \forall$, the function † simply replaces every atom P by $\neg\neg P$. It is easy to show that A and $A^\dagger$ are classically equivalent. The following lemma shows the significance of this function.

Lemma 15 *For every formula A built only from $\supset, \wedge, \neg, \forall$, the sequent $\neg\neg A^\dagger \vdash A^\dagger$ is provable in the system $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$.*

Proof. Since $\neg\neg\neg A \equiv \neg A$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$, the sequent $\neg\neg\neg\neg P \equiv \neg\neg P$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$ for every atom P, and thus the result follows from the definition of $A^\dagger$ and Lemma 14. □

Actually, we can state a slightly more general version of Lemma 15, based on the observation that $\neg\neg\neg A \equiv \neg A$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$.

Lemma 16 *For every formula A built only from $\supset, \wedge, \neg, \forall$ and where every atomic subformula occurs negated (except $\perp$), the sequent $\neg\neg A^\dagger \vdash A^\dagger$ is provable in the system $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$.*

The formulae of the kind mentioned in Lemma 16 are called *negative* formulae. The following lemma shows that if we use double-negation, then $\vee$ and $\exists$ are definable intuitionistically from the connectives $\wedge, \neg, \forall$.

Lemma 17 *The following formulae are provable in* $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$*:*

$$\begin{aligned}\neg\neg(A \vee B) &\equiv \neg(\neg A \wedge \neg B)\\ \neg\neg\exists x A &\equiv \neg\forall x \neg A.\end{aligned}$$

Proof. We give a proof of the sequent $\neg\neg\exists x A \vdash \neg\forall x \neg A$, leaving the others as exercises.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A[y/x] \vdash A[y/x]}{\neg A[y/x], A[y/x] \vdash}}{\forall x \neg A, A[y/x] \vdash}}{\forall x \neg A, \exists x A \vdash}}{\forall x \neg A \vdash \neg\exists x A}}{\neg\neg\exists x A, \forall x \neg A \vdash}}{\neg\neg\exists x A \vdash \neg\forall x \neg A}$$

where y is a new variable. □

We are now ready to prove the main lemma about the double-negation translation. The correctness of many embeddings of classical logic into intuitionistic logic follows from this lemma, including those due to Kolmogorov, Gödel, and Gentzen.

Lemma 18 *Let* $\Gamma \vdash B_1, \ldots, B_n$ *be any first-order sequent containing formulae made only from* $\supset, \wedge, \neg$*, and* $\forall$*. If* $\Gamma \vdash B_1, \ldots, B_n$ *is provable in* $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$ *then its translation* $\Gamma^\dagger \vdash \neg(\neg B_1^\dagger \wedge \ldots \wedge \neg B_n^\dagger)$ *is provable in* $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$*. In particular, if* B *is provable in* $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$*, then* $B^\dagger$ *is provable in* $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$*.*

Proof. First, we prove that if $\Gamma \vdash B_1, \ldots, B_n$ is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$ then $\Gamma^\dagger \vdash B_1^\dagger, \ldots, B_n^\dagger$ is also provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$. This is done by a simple induction on proofs. Next, we prove that $\neg\neg\Gamma^\dagger \vdash \neg(\neg B_1^\dagger \wedge \ldots \wedge \neg B_n^\dagger)$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$. The only obstacle to Lemma 13 is the use of the ($\forall$: *right*)-rule. However, we have seen in the discussion following Lemma 13 that the problem is overcome for formulae such that $\neg\neg A \vdash A$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$. But this is the case by Lemma 15 (which itself is a direct consequence of Lemma 14), since we are now considering formulae of the form $A^\dagger$. Since $B \vdash \neg\neg B$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$ for any B, using cuts on the premises in $\neg\neg\Gamma^\dagger$, we obtain a proof of $\Gamma^\dagger \vdash \neg(\neg B_1^\dagger \wedge \ldots \wedge \neg B_n^\dagger)$ in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$. In the special case where $n = 1$ and Γ is empty, we have shown that $\vdash \neg\neg B^\dagger$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$, and using Lemma 15, we obtain that $A^\dagger$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$. □

It is trivial that the converse of Lemma 18 holds (since $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$ is a subsystem of $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$). As a corollary of Lemma 18, observe that for negative formulae (defined in

Lemma 16), A is provable in $\mathcal{G}_c^{\supset,\wedge,\neg,\forall}$ iff A is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$. This is because for a negative formula A, all atoms appears negated, and thus $A \equiv A^\dagger$ is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$.

We now define several translations of classical logic into intuitionistic logic.

Definition 27 *The function $^\circ$ (due to Gentzen) is defined as follows:*

$$\begin{aligned} A^\circ &= \neg\neg A, \quad \textit{if } A \textit{ is atomic}, \\ (\neg A)^\circ &= \neg A^\circ, \\ (A \wedge B)^\circ &= (A^\circ \wedge B^\circ), \\ (A \supset B)^\circ &= (A^\circ \supset B^\circ), \\ (A \vee B)^\circ &= \neg(\neg A^\circ \wedge \neg B^\circ), \\ (\forall x\, A)^\circ &= \forall x\, A^\circ, \\ (\exists x\, A)^\circ &= \neg\forall x \neg A^\circ. \end{aligned}$$

The function * (due to Gödel) is defined as follows:

$$\begin{aligned} A^* &= \neg\neg A, \quad \textit{if } A \textit{ is atomic}, \\ (\neg A)^* &= \neg A^*, \\ (A \wedge B)^* &= (A^* \wedge B^*), \\ (A \supset B)^* &= \neg(A^* \wedge \neg B^*), \\ (A \vee B)^* &= \neg(\neg A^* \wedge \neg B^*), \\ (\forall x\, A)^* &= \forall x\, A^*, \\ (\exists x\, A)^* &= \neg\forall x \neg A^*. \end{aligned}$$

The function $^\kappa$ (due to Kolmogorov) is defined as follows:

$$\begin{aligned} A^\kappa &= \neg\neg A, \quad \textit{if } A \textit{ is atomic}, \\ (\neg A)^\kappa &= \neg A^\kappa, \\ (A \wedge B)^\kappa &= \neg\neg(A^\kappa \wedge B^\kappa), \\ (A \supset B)^\kappa &= \neg\neg(A^\kappa \supset B^\kappa), \\ (A \vee B)^\kappa &= \neg\neg(A^\kappa \vee B^\kappa), \\ (\forall x\, A)^\kappa &= \neg\neg\forall x\, A^\kappa, \\ (\exists x\, A)^\kappa &= \neg\neg\exists x\, A^\kappa. \end{aligned}$$

By Lemma 17 and Lemma 14, it is easy to show that for any sequent $\Gamma \vdash B_1, \ldots, B_n$, the sequent

$$\Gamma^\circ \vdash \neg(\neg B_1^\circ \wedge \ldots \wedge \neg B_n^\circ)$$

is provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$ iff

$$\Gamma^* \vdash \neg(\neg B_1^* \wedge \ldots \wedge \neg B_n^*)$$

is provable iff

$$\Gamma^{\kappa} \vdash \neg(\neg B_1^{\kappa} \wedge \ldots \wedge \neg B_n^{\kappa})$$

is provable. Furthermore, it is easily shown that $A \equiv A^{\circ}$, $A \equiv A^{*}$, and $A \equiv A^{\kappa}$, are provable classically.

Theorem 9 *For any sequent $\Gamma \vdash B_1, \ldots, B_n$, if $\Gamma \vdash B_1, \ldots, B_n$ is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$, then the sequents $\Gamma^{\circ} \vdash \neg(\neg B_1^{\circ} \wedge \ldots \wedge \neg B_n^{\circ})$, $\Gamma^{*} \vdash \neg(\neg B_1^{*} \wedge \ldots \wedge \neg B_n^{*})$, and $\Gamma^{\kappa} \vdash \neg(\neg B_1^{\kappa} \wedge \ldots \wedge \neg B_n^{\kappa})$, are provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$. In particular, if A is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$, then A°, A^{*}, and A^{κ}, are provable in $\mathcal{G}_i^{\supset,\wedge,\neg,\forall}$.*

Proof. We simply have to observe that the translation $^{\circ}$ is in fact the composition of two functions: the first one $^{\bullet}$ is defined as in Definition 27, *except* that atoms remain unchanged, and the second function is just †. This translation has the property that $A^{\bullet}$ only contains the connectives $\supset$, $\wedge$, $\neg$, and $\forall$. Furthermore, it is easily shown that $\Gamma \vdash B_1, \ldots, B_n$ is provable in $\mathcal{G}_c^{\supset,\wedge,\vee,\neg,\forall,\exists}$ iff $\Gamma^{\bullet} \vdash B_1^{\bullet}, \ldots, B_n^{\bullet}$ is. Therefore, Lemma 18 applies to $\Gamma^{\bullet} \vdash B_1^{\bullet}, \ldots, B_n^{\bullet}$, and we obtain the desired result. □

It is trivial that the converse of Theorem 9 holds.

We shall now discuss another translation of classical logic into intuitionistic logic due to Girard [9]. Girard has pointed out that the usual double-negation translations have some rather undesirable properties:

(1) They are not compatible with substitution. Indeed, the translation $A[B/P]^{*}$ of $A[B/P]$ is not equal to $A^{*}[B^{*}/P]$ in general, due to the application of double negations to atoms.

(2) Negation is not involutive. For instance, $A[B/P]^{*}$ and $A^{*}[B^{*}/P]$ are related through the erasing of certain double negations (passing from $\neg\neg\neg P$ to $\neg P$), but this erasing is not harmless.

(3) Disjunction is not associative. For example, if $A \vee B$ is translated as $\neg(\neg A \wedge \neg B)$, then $(A \vee B) \vee C$ is translated as $\neg(\neg(\neg(\neg A \wedge \neg B)) \wedge \neg C)$, and $A \vee (B \vee C)$ is translated as $\neg(\neg A \wedge \neg(\neg(\neg B \wedge \neg C)))$.

Girard has discovered a translation which does not suffer from these defects, and this translation also turns out to be quite economical in the number of negation signs introduced [9]. The main idea is assign a *sign* or *polarity* (+ or −) to every formula. Roughly speaking, a positive literal P (where P is an atom) is a formula of polarity +, a negative literal $\neg P$ is a formula of polarity −, and to determine the polarity of a compound formula, we combine its polarities as if they were truth values, except that + corresponds to **false**, − corresponds to **true**, existential formulae are always positive, and universal formulae are always negative. Given a sequent $\Gamma \vdash \Delta$, the idea is that right-rules have to be converted to left-rules, and in order to do this we need to move formulae in Δ to the lefthand side of the sequent. The new

twist is that formulae in Δ will be treated differently according to their polarity. Every formula A of polarity $-$ in Δ is in fact of the form $\neg B$, and it will be transferred to the lefthand side as B (and not as $\neg\neg B$), and every formula A of polarity + in Δ will be transferred to the lefthand side as $\neg A$. The translation is then completely determined if we add the obvious requirement that the translation of a classically provable sequent should be intuitionistically provable. Let us consider some typical cases.

Case 1. The last inference is

$$\frac{\Gamma \vdash \neg C, \neg D}{\Gamma \vdash \neg C \vee \neg D}$$

where C and D are positive. The sequent $\Gamma \vdash \neg C, \neg D$ is translated as $\Gamma, C, D \vdash$, and we have the inference

$$\frac{\Gamma, C, D \vdash}{\Gamma, C \wedge D \vdash}$$

It is thus natural to translate $\neg C \vee \neg D$ as $\neg(C \wedge D)$, since then $C \wedge D$ will be placed on the lefthand side (because $\neg(C \wedge D)$ is negative).

Case 2. The last inference is

$$\frac{\Gamma \vdash C, D}{\Gamma \vdash C \vee D}$$

where C and D are positive. The sequent $\Gamma \vdash C, D$ is translated as $\Gamma, \neg C, \neg D \vdash$, and we have the inference

$$\frac{\Gamma, \neg C, \neg D \vdash}{\Gamma, \neg C \wedge \neg D \vdash}$$

This time, we would like to translate $C \vee D$ as $C \vee D$ (since $C \vee D$ is positive), so that $\neg(C \vee D)$ is placed on the lefthand side of the sequent. This is indeed legitimate because $\neg(C \vee D) \equiv \neg C \wedge \neg D$ is provable intuitionistically.

Case 3. The last inference is

$$\frac{\Gamma \vdash C}{\Gamma \vdash \forall x C}$$

where C is positive. The sequent $\Gamma \vdash C$ is translated as $\Gamma, \neg C \vdash$, and we have the inference

$$\frac{\Gamma, \neg C \vdash}{\Gamma, \exists x \neg C \vdash}$$

We translate $\forall x C$ as $\neg\exists x \neg C$, so that $\exists x \neg C$ is placed on the lefthand side of the sequent.

Case 4. The last inference is

$$\frac{\Gamma \vdash \neg C}{\Gamma \vdash \forall x \neg C}$$

where C is positive. The sequent $\Gamma \vdash \neg C$ is translated as $\Gamma, C \vdash$, and we have the inference

$$\frac{\Gamma, C \vdash}{\Gamma, \exists x C \vdash}$$

We translate $\forall x \neg C$ as $\neg \exists x C$, so that $\exists x C$ is placed on the lefthand side of the sequent.

Case 5. The last inference is

$$\frac{\Gamma \vdash C[t/x]}{\Gamma \vdash \exists x C}$$

where C is positive. The sequent $\Gamma \vdash C[t/x]$ is translated as $\Gamma, \neg C[t/x] \vdash$, and we have the inference

$$\frac{\Gamma, \neg C[t/x] \vdash}{\Gamma, \forall x \neg C \vdash}$$

We would like to translate $\exists x C$ as $\exists x C$, so that $\neg \exists x C$ is placed on the lefthand side of the sequent. This is possible because $\neg \exists x C \equiv \forall x \neg C$ is provable intuitionistically.

Case 6. The last inference is

$$\frac{\Gamma \vdash \neg C[t/x]}{\Gamma \vdash \exists x \neg C}$$

where C is positive. The sequent $\Gamma \vdash \neg C[t/x]$ is translated as $\Gamma, C[t/x] \vdash$. We would like to translate $\exists x \neg C$ as $\exists x \neg C$, so that $\neg \exists x \neg C$ is placed on the lefthand side of the sequent. This is possible because $\neg \exists x \neg C \equiv \forall x \neg \neg C$ is provable intuitionistically, and we have the sequence of inferences

$$\frac{\dfrac{\dfrac{\Gamma, C[t/x] \vdash}{\Gamma \vdash \neg C[t/x]}}{\Gamma, \neg \neg C[t/x] \vdash}}{\Gamma, \forall x \neg \neg C[t/x] \vdash}$$

Note that it was necessary to first double-negate $C[t/x]$. This is because $\neg \exists x \neg C \equiv \forall x \neg \neg C$ is provable intuitionistically, but $\neg \exists x \neg C \equiv \forall x C$ is not.

Case 7. The last inference is

$$\frac{\Gamma \vdash C \quad \Gamma \vdash D}{\Gamma \vdash C \wedge D}$$

where C, D are positive. The sequents $\Gamma \vdash C$ and $\Gamma \vdash D$ are translated as $\Gamma, \neg C \vdash$ and $\Gamma, \neg D \vdash$. Since $C \wedge D$ is positive, we would like to translate $C \wedge D$ as $C \wedge D$, so that $\neg(C \wedge D)$ is placed on the lefthand side of the sequent. This is possible because $\neg(\neg \neg C \wedge \neg \neg D) \equiv \neg(C \wedge D)$ is provable intuitionistically, and we have the sequence of inferences

$$\dfrac{\dfrac{\dfrac{\Gamma, \neg C \vdash}{\Gamma \vdash \neg\neg C} \qquad \dfrac{\Gamma, \neg D \vdash}{\Gamma \vdash \neg\neg D}}{\Gamma \vdash \neg\neg C \wedge \neg\neg D}}{\Gamma, \neg(\neg\neg C \wedge \neg\neg D) \vdash}$$

Case 8. The last inference is

$$\dfrac{\Gamma \vdash \neg C \quad \Gamma \vdash D}{\Gamma \vdash \neg C \wedge D}$$

where C, D are positive. The sequents $\Gamma \vdash \neg C$ and $\Gamma \vdash D$ are translated as $\Gamma, C \vdash$ and $\Gamma, \neg D \vdash$. Since $\neg C \wedge D$ is positive, we would like to translate $\neg C \wedge D$ as $\neg C \wedge D$, so that $\neg(\neg C \wedge D)$ is placed on the lefthand side of the sequent. This is possible because $\neg(\neg C \wedge \neg\neg D) \equiv \neg(\neg C \wedge D)$ is provable intuitionistically, and we have the sequence of inferences

$$\dfrac{\dfrac{\dfrac{\Gamma, C \vdash}{\Gamma \vdash \neg C} \qquad \dfrac{\Gamma, \neg D \vdash}{\Gamma \vdash \neg\neg D}}{\Gamma \vdash \neg C \wedge \neg\neg D}}{\Gamma, \neg(\neg C \wedge \neg\neg D) \vdash}$$

Case 9. The last inference is

$$\dfrac{\Gamma \vdash \neg C \quad \Gamma \vdash \neg D}{\Gamma \vdash \neg C \wedge \neg D}$$

where C, D are positive. The sequents $\Gamma \vdash \neg C$ and $\Gamma \vdash \neg D$ are translated as $\Gamma, C \vdash$ and $\Gamma, D \vdash$, and we have the inference

$$\dfrac{\Gamma, C \vdash \quad \Gamma, D \vdash}{\Gamma, C \vee D \vdash}$$

We translate $\neg C \wedge \neg D$ as $\neg(C \vee D)$, so that $C \vee D$ is placed on the lefthand side of the sequent.

Considering all the cases, we arrive at the following tables defining the Girard translation $\widehat{A}$ of a formula.

Definition 28 *Given any formula A, its sign (polarity) and its Girard-translation $\widehat{A}$ are given by the following tables:*

*If $A = P$ where P is an atom, including the constants $\top$ (***true***) and $\bot$ (***false***), then $sign(A) = +$ and $\widehat{A} = A$, and if A is a compound formula then*

Girard's $\neg\neg$-Translation				
A	B	$A \wedge B$	$A \vee B$	$A \supset B$
$+, C$	$+, D$	$+, C \wedge D$	$+, C \vee D$	$-, \neg(C \wedge \neg D)$
$+, C$	$-, \neg D$	$+, C \wedge \neg D$	$-, \neg(\neg C \wedge D)$	$-, \neg(C \wedge D)$
$-, \neg C$	$+, D$	$+, \neg C \wedge D$	$-, \neg(C \wedge \neg D)$	$+, C \vee D$
$-, \neg C$	$-, \neg D$	$-, \neg(C \vee D)$	$-, \neg(C \wedge D)$	$-, \neg(\neg C \wedge D)$

Girard's $\neg\neg$-Translation			
A	$\forall x\, A$	$\exists x\, A$	$\neg A$
$+, C$	$-, \neg\exists x \neg C$	$+, \exists x C$	$-, \neg C$
$-, \neg C$	$-, \neg\exists x C$	$+, \exists x \neg C$	$+, C$

Given a formula A, we define its translation $\overline{A}$ as follows:

$$\overline{A} = \begin{cases} \neg\widehat{A} & \text{if } sign(A) = +, \\ B & \text{if } sign(A) = - \text{ and } \widehat{A} = \neg B. \end{cases}$$

Then, a sequent $\Gamma \vdash \Delta$ is translated into the sequent $\Gamma, \overline{\Delta} \vdash$.

We have the following theorem.

Theorem 10 *Given any classical sequent $\Gamma \vdash \Delta$, if $\Gamma \vdash \Delta$ is provable classically, then its translation $\Gamma, \overline{\Delta} \vdash$ is provable intuitionistically.*

Proof. By induction on the structure of proofs. We have already considered a number of cases in the discussion leading to the tables of Definition 28. As an auxiliary result, we need to show that the following formulae are provable intuitionistically:

$$\begin{aligned} \neg(C \vee D) &\equiv \neg C \wedge \neg D, \\ \neg(\neg\neg C \wedge \neg\neg D) &\equiv \neg(C \wedge D), \\ \neg(\neg C \wedge \neg\neg D) &\equiv \neg(\neg C \wedge D), \\ \neg(\neg\neg C \wedge \neg D) &\equiv \neg(C \wedge \neg D), \\ \neg\exists x C &\equiv \forall x \neg C, \\ \neg\exists x \neg C &\equiv \forall x \neg\neg C. \end{aligned}$$

We leave the remaining cases as an exercise. □

Observe that a formula A of any polarity can be made into an equivalent formula of polarity +, namely $A^+ = A \wedge \top$, or an equivalent formula of polarity $-$, namely $A^- = A \vee \neg\top$. The Girard-translation has some nice properties, and the following lemma lists some of them [9].

Lemma 19 *The translation $A \mapsto \widehat{A}$ given in Definition 28 is compatible with substitutions respecting polarities. Furthermore, it satisfies a number of remarkable identities:*

(i) Negation is involutive: $\neg\neg A \equiv A$.

(ii) De Morgan identities: $\neg(A \wedge B) \equiv \neg A \vee \neg B$; $\neg(A \vee B) \equiv \neg A \wedge \neg B$; $A \supset B \equiv \neg A \vee B$; $\neg\forall \boldsymbol{x} A \equiv \exists \boldsymbol{x} \neg A$; $\neg\exists \boldsymbol{x} A \equiv \forall \boldsymbol{x} \neg A$.

(iii) Associativity of $\wedge$ *and* $\vee$*; as a consequence,* $(A \wedge B) \supset C \equiv A \supset (B \supset C)$*, and* $A \supset (B \vee C) \equiv (A \supset B) \vee C$.

(iv) Neutrality identities: $A \vee \bot \equiv A$; $A \wedge \neg\bot \equiv A$.

(v) Commutativity of $\wedge$ *and* $\vee$ *(as a consequence,* $A \supset B \equiv \neg B \supset \neg A$*).*

(vi) Distributivity identities with restriction on polarities: $A \wedge (P \vee Q) \equiv (A \wedge P) \vee (A \wedge Q)$*;* $A \vee (L \wedge M) \equiv (A \vee L) \wedge (A \vee M)$ *(where* P, Q *are positive, and* L, M *negative).*

(vii) Idempotency identities: $P^+ \equiv P$ *where* P *is positive;* $N^- \equiv N$ *where* N *is negative; as a consequence,* $A^{++} \equiv A^+$ *and* $A^{--} \equiv A^-$.

(viii) Quantifier isomophisms: $A \wedge \exists \boldsymbol{x} P \equiv \exists \boldsymbol{x}(A \wedge P)$ *if* $\boldsymbol{x}$ *is not free in* A *and* P *is positive;;* $A \vee \forall \boldsymbol{x} N \equiv \forall \boldsymbol{x}(A \vee N)$ *if* $\boldsymbol{x}$ *is not free in* A *and* N *is negative;*

Proof. The proof is quite straightforward, but somewhat tedious. Because of the polarities, many cases have to be considered. Some cases are checked in Girard [9], and the others can be easily verified. □

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